Section 10.5

1. For the following power series, find the radius and interval of convergence.

   a.) $\sum_{n=0}^{\infty} \frac{n^2 x^n}{3^n}$

For the power series $\sum_{n=1}^{\infty} c_n (x-a)^n$ the center of the series is a.

we want to find the interval of convergence = the interval that contains all values of x for which the series converges, denoted by I.

case 1: $I = \{a\}$ $\boxed{R=0}$

case 2: $I = (-\infty, \infty)$ $\boxed{R=\infty}$

case 3: $I = (a-R, a+R)$

R = radius of convergence.

must also test endpoints for convergence.

$a-R \quad \overbrace{a \quad a+R}^{R}$

Section 10.5

1. For the following power series, find the radius and interval of convergence.

   a.) $\sum_{n=0}^{\infty} \frac{n^2 x^n}{3^n}$

centered at 0.

Step 1: Apply Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$\lim_{n\to\infty} \left| \frac{(n+1)^2 x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n^2 x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 x \cdot \cancel{x^n}}{3 \cdot \cancel{3^n}} \cdot \frac{\cancel{3^n}}{n^2 \cancel{x^n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x}{3} \cancel{\frac{(n+1)^2}{n^2}}^{1} \right| = \left| \frac{x}{3} \right| < 1 \longrightarrow \frac{1}{3}|x| < 1$$

$$|x| < \underline{\underline{3}}_{R}$$

div   -3   conv   0   3   div

center

Test endpoints for convergence: plug $x = \pm 3$ into $\sum_{n=0}^{\infty} \frac{n^2 x^n}{3^n}$

$x=3$: $\sum_{n=0}^{\infty} \frac{n^2 3^n}{3^n} = \sum_{n=0}^{\infty} n^2$ T.D. $\lim_{n\to\infty} n^2 \neq 0$

$\sum_{n=0}^{\infty} n^2$ diverges by

$x=-3$: $\sum \frac{n^2 (-3)^n}{} = \sum (-1)^n n^2$ $n^2 \neq 0$

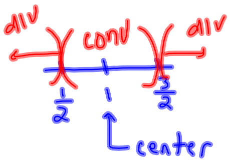
b.) $\sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{n^2+2}$ centered at 1.

RT: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}(x-1)^{n+1}}{(n+1)^2+2} \cdot \frac{n^2+2}{2^n (x-1)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{2 \cdot \cancel{2^n}(x-1)\cancel{(x-1)^n}}{(n+1)^2+2} \cdot \frac{n^2+2}{\cancel{2^n}\cancel{(x-1)^n}} \right| = \lim_{n\to\infty} \left| \frac{2(x-1)(n^2+2)}{(n+1)^2+2} \right|$ $\to 1$

$= |2(x-1)| < 1 \to 2|x-1| < 1$

$\to |x-1| < \frac{1}{2}$ $\equiv R$



Test endpoints for convergence by plugging $x = \frac{1}{2}$ + $\frac{3}{2}$ into b.) $\sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{n^2+2}$: $x = \frac{3}{2}$ $\sum_{n=1}^{\infty} \frac{2^n \left(\frac{1}{2}\right)^n}{n^2+2}$

$= \sum_{n=1}^{\infty} \frac{1}{n^2+2} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$

Larger converges by p-series $p=2$.

Smaller converges by Comparison Test

include $x = \frac{3}{2}$ on interval of conv.

$x = \frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{2^n \left(-\frac{1}{2}\right)^n}{n^2+2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2}$

AST show: $\left\{ \frac{1}{n^2+2} \right\}$ decreases to zero.

decreases: $\frac{1}{(n+1)^2+2} \le \frac{1}{n^2+2}$

$a_{n+1} \le a_n$ ✓

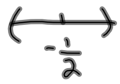
to zero: $\lim_{n\to\infty} \frac{1}{n^2+2} = 0$ ✓

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2}$ converges by AST.

include $x = \frac{1}{2}$ on interval of conv.

$\boxed{I = \left[\frac{1}{2}, \frac{3}{2}\right] \quad R = \frac{1}{2}}$

c.) $\sum_{n=1}^{\infty} \frac{(-1)^n(2x+1)^n}{\sqrt{n}}$ center: $x = -\frac{1}{2}$



RT: $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(2x+1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-1)^n(2x+1)^n} \right|$

$$\lim_{n\to\infty} \left| \frac{(-1)\cancel{(-1)^n}(2x+1)\cancel{(2x+1)^n}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\cancel{(-1)^n}\cancel{(2x+1)^n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{-1(2x+1)\sqrt{n}}{\sqrt{n+1}} \right| = \left| -1(2x+1) \right|$$ $|ab| = |a||b|$

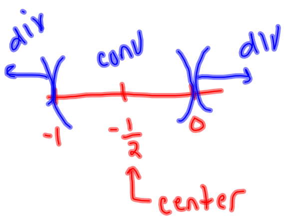
$\frac{\sqrt{n}}{\sqrt{n+1}} \to 1$

$= |2x+1| < 1$

$= \left| 2\left(x+\frac{1}{2}\right) \right| < 1$

$\left| x + \frac{1}{2} \right| < \frac{1}{2}$

$\left| x - \left(-\frac{1}{2}\right) \right| < \underbrace{\frac{1}{2}}_{R}$



Test endpoints by plugging $x=-1$ & $x=0$ into c.) $\sum_{n=1}^{\infty} \frac{(-1)^n(2x+1)^n}{\sqrt{n}}$

$x=0$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by AST since $\left\{ \frac{1}{\sqrt{n}} \right\}$ decreases to zero (show it!)

include $x=0$ in interval.

$x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

p-series $p = \frac{1}{2} < 1$ diverges

do not include $x=-1$ in interval.

$$\boxed{I = (-1, 0] \quad R = \frac{1}{2}}$$

d.) $\sum_{n=0}^{\infty} \frac{n!(x+2)^{n-1}}{5^{n-1}}$ center: -2.

RT: $\lim_{n\to\infty} \left| \frac{(n+1)!\,(x+2)^{n}}{5^{n}} \cdot \frac{5^{n-1}}{n!\,(x+2)^{n-1}} \right|$

$\lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}\cdot(x+2)\cancel{(x+2)^{n-1}}}{5\cdot\cancel{5^{n-1}}} \cdot \frac{\cancel{5^{n-1}}}{\cancel{n!}\cancel{(x+2)^{n-1}}} \right|$

$\lim_{n\to\infty} \left| \frac{x+2}{5}(n+1) \right| = \infty$ unless $x = -2$.

$\boxed{I = \{-2\} \quad R = 0}$

e.) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{(2n+1)!}$ center is 0.

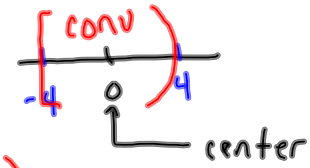
$\lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-3)^n x^n} \right|$

$\lim_{n\to\infty} \left| \frac{-3\cancel{(-3)^n}\, x\cdot\cancel{x^n}}{(2n+3)(2n+2)\cancel{(2n+1)!}} \cdot \frac{\cancel{(2n+1)!}}{\cancel{(-3)^n}\,\cancel{x^n}} \right|$

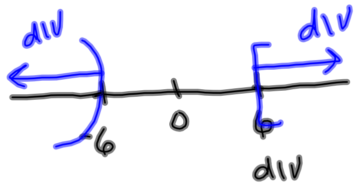
$\lim_{n\to\infty} \left| \frac{-3x}{(2n+3)(2n+2)} \right| = 0$ for all x

$\boxed{I = (-\infty, \infty) \quad R = \infty}$

2. Suppose it is known that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. On what interval(s) are we guaranteed convergence? On what interval(s) are we guaranteed divergence?



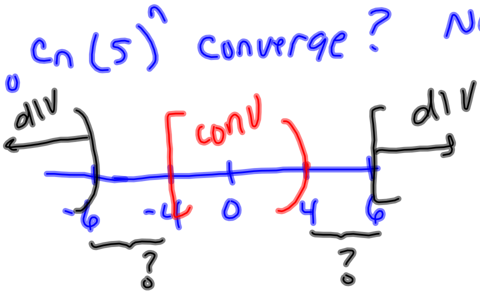
guaranteed convergence on $[-4, 4)$



guaranteed divergence on

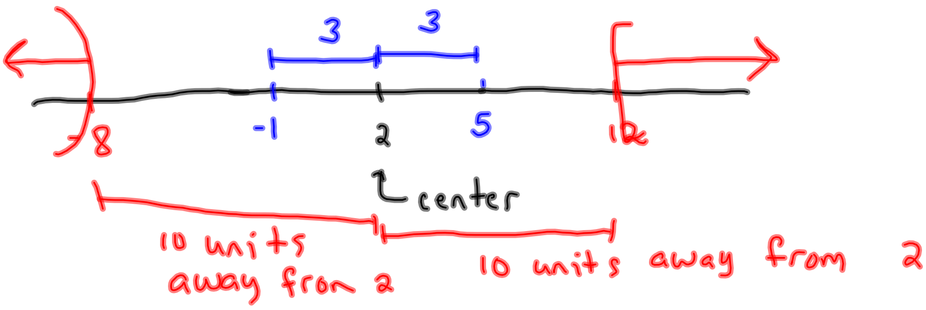
$(-\infty, -6) \cup [6, \infty)$

does $\sum_{n=0}^{\infty} c_n (5)^n$ converge? not enough information



3. Suppose it is known that $\sum_{n=0}^{\infty} c_n (x-2)^n$ converges when $x = 5$ and diverges when $x = 12$. On what interval(s) are we guaranteed convergence? On what interval(s) are we guaranteed divergence?

convergence : $(-1, 5]$



divergence: $(-\infty, -8)$,

$[12, \infty)$

Section 10.6

4. Express the following functions as a power series. Identify the radius and interval of convergence.

a.) $f(x) = \frac{1}{1-x}$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ if } |x|<1$$

$\boxed{R=1}$

test endpoints: $x=1$ $\sum_{n=0}^{\infty} (1)^n$ div

$x=-1$ $\sum_{n=0}^{\infty} (-1)^n$ div

$$\boxed{\begin{array}{l} I=(-1,1) \\ R=1 \\ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \end{array}}$$

b.) $f(x) = \frac{1}{1-5x}$ $\boxed{= \sum_{n=0}^{\infty} (5x)^n}$,

where $|5x|<1 \rightarrow |x| < \frac{1}{5}$ $R = \frac{1}{5}$

$I = (-\frac{1}{5}, \frac{1}{5})$

c.) $f(x) = \frac{-3}{1+4x^2}$

Form "$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$"

$$= -3\left(\frac{1}{1-(-4x^2)}\right)$$

$$= -3 \sum_{n=0}^{\infty} (-4x^2)^n$$

$$= -3 \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}$$

$$= \boxed{3 \sum_{n=0}^{\infty} (-1)^{n+1} 4^n x^{2n}}$$

$|-4x^2| < 1$

$4|x^2| < 1$

$|x^2| < \frac{1}{4}$

$|x| < \frac{1}{2}$

$$\boxed{\begin{array}{l} R=\frac{1}{2} \\ I=(-\frac{1}{2}, \frac{1}{2}) \end{array}}$$

d.) $f(x) = \frac{3x^2}{9-x}$

$$= \frac{3x^2}{9(1-\frac{x}{9})}$$

$$= \frac{x^2}{3} \frac{1}{1-\frac{x}{9}}$$

$$= \frac{x^2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{9}\right)^n,$$ where $\left|\frac{x}{9}\right| < 1$

$|x| < 9$

$R = 9$

$I = (-9, 9)$

$$= \frac{x^2}{3} \sum_{n=0}^{\infty} \frac{x^n}{9^n}$$

$$= \frac{x^2}{3} \sum_{n=0}^{\infty} \frac{x^n}{3^{2n}}$$

$$\frac{x^2}{3} = \sum \frac{x^{n+2}}{\ldots} \text{ if } -9 < x < 9$$

e.) $f(x) = \ln(x+4)$

$$\frac{d}{dx}\ln(x+4) = \frac{1}{x+4}$$

$$= \frac{1}{4\left(1+\frac{x}{4}\right)} = \frac{1}{4}\,\frac{1}{1-\underbrace{\left(\frac{-x}{4}\right)}_{r}}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n, \quad \left|-\frac{x}{4}\right|<1$$

$$\boxed{|x|<4}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^n}$$

$$\int \frac{d}{dx}\ln(x+4)\,dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^{n+1}}\,dx$$

$$\ln(x+4) = C + \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\frac{x^{n+1}}{n+1}, \quad |x|<4$$

let $x=0$ to solve for c:

$$\ln 4 = C + \cancel{\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\frac{0^{n+1}}{n+1}} \to 0 \qquad \boxed{C = \ln 4}$$

$$\boxed{\ln(x+4) = \ln 4 + \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\frac{x^{n+1}}{n+1}, \quad |x|<4}$$

f.) $f(x) = x\ln(x+4)$ multiply ☆ by "x"

$$x\ln(x+4) = x\left(\ln 4 + \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\frac{x^{n+1}}{n+1}\right)$$

$$= \boxed{x\ln 4 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{4^{n+1}(n+1)}}$$

g.) $f(x) = \arctan(2x^3)$

$$\frac{d}{dx}\arctan(2x^3) = \frac{6x^2}{1+4x^6}$$

$$= 6x^2\left(\frac{1}{1-(-4x^6)}\right)$$

$$= 6x^2\sum_{n=0}^{\infty}(-4x^6)^n \qquad |-4x^6|<1$$

$$|x| < \sqrt[6]{\frac{1}{4}}$$

$$= 6x^2\sum_{n=0}^{\infty}(-1)^n 4^n x^{6n}$$

$$= 3\cdot 2\, x^2 \sum_{n=0}^{\infty}(-1)^n 2^{2n} x^{6n}$$

$$\int \frac{d}{dx}\arctan(2x^3)\,\cancel{dx} = \int 3\sum_{n=0}^{\infty}(-1)^n 2^{2n+1} x^{6n+2}\,dx$$

$$\arctan(2x^3) = \cancel{C}^{\,0} + 3\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1} x^{6n+3}}{6n+3}$$

let $x=0$ to find C $\qquad \arctan 0 = C + \sum 0$

$$\boxed{C=0}$$

side Fact →

$$\frac{d}{dx}\sum_{\boxed{n=0}}^{\infty} c_n x^n = \frac{d}{dx}\left[c_0 + c_1x + c_2x^2 + c_3x^3 + \dots\right]$$

note!

$$= c_1 + 2c_2x + 3c_3x^2 + \dots$$

$$= \sum_{\boxed{n=1}}^{\infty} c_n n x^{n-1}$$

h.) $f(x) = \dfrac{1}{(1-2x)^2}$

$$\int\frac{dx}{(1-2x)^2} \qquad u = 1-2x \qquad du = -2\,dx$$

$$-\frac{1}{2}\int\frac{du}{u^2} = -\frac{1}{2}\cdot\frac{-1}{u} = \frac{1}{2}\cdot\frac{1}{1-2x}$$

$$\int\frac{dx}{(1-2x)^2} = \frac{1}{2}\cdot\frac{1}{1-2x}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(2x)^n \qquad |2x|<1 \qquad \boxed{|x|<\frac{1}{2}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}2^n x^n$$

$$\frac{d}{dx}\int\frac{dx}{(1-2x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty}2^{n-1}x^n$$

$$\frac{1}{(1-2x)^2} = \sum_{n=1}^{\infty}2^{n-1}\,n\,x^{n-1}$$

$$= \sum_{n=0}^{\infty}2^n(n+1)x^n$$

5. Express $\int_0^{0.1} \frac{1}{1+x^5}\,dx$ as an infinite series. Use the sum of the first 3 terms of this series to approximate $\int_0^{0.1} \frac{1}{1+x^5}\,dx$. Estimate the error.

$$\int_0^{\frac{1}{10}} \frac{dx}{1+x^5} = \int_0^{\frac{1}{10}} \frac{dx}{1-(-x^5)}$$

$$= \int_0^{\frac{1}{10}} \sum_{n=0}^{\infty} (-x^5)^n\,dx$$

$$= \int_0^{\frac{1}{10}} \sum_{n=0}^{\infty} (-1)^n x^{5n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{5n+1} \Bigg|_0^{\frac{1}{10}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{10}\right)^{5n+1}}{5n+1} - \sum (0)$$

$$\int_0^{\frac{1}{10}} \frac{dx}{1+x^5} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(5n+1)10^{5n+1}}$$

$$\int_0^{\frac{1}{10}} \frac{dx}{1+x^5} \approx \sum_{n=0}^{2} \frac{(-1)^n}{(5n+1)10^{5n+1}}$$

$$S_2 = \frac{1}{10} - \frac{1}{6(10^6)} + \frac{1}{(11)10^{11}}$$

$$|R_2| \le |a_3| = \frac{1}{(16)10^{16}}$$