Section 8.1

1. $\int x^2 \ln x \, dx$

$u = \ln x$, $dv = x^2 dx$

$du = \frac{1}{x} dx$, $v = \frac{x^3}{3}$

Int by parts formula

$\int u\,dv = uv - \int v\,du$

you choose the u and dv

u =

I = inverse trig

L = $\ln x$

A = algebraic

T = trig

E = e^x

$$\int x^2 \ln x \, dx = uv - \int v\,du$$

$$= (\ln x)\left(\frac{x^3}{3}\right) - \int \frac{x^3}{3} \cdot \frac{1}{x}\, dx$$

$$= (\ln x)\left(\frac{x^3}{3}\right) - \frac{1}{3}\int x^2\, dx$$

$$= (\ln x)\left(\frac{x^3}{3}\right) - \frac{1}{3}\,\frac{x^3}{3} + C$$

2. $\int_0^1 \frac{x}{e^{3x}}\,dx$

I
L
$\boxed{A} \rightarrow x,\ x^2,\ x^3$
T
$\boxed{E}$

$$\underline{\underline{\int_0^1}}\ x e^{-3x}\,dx$$

$u = x, \quad dv = e^{-3x}\,dx$

$du = dx, \quad v = -\frac{1}{3}e^{-3x}$

$$\int_0^1 x e^{-3x}\,dx = uv\Big|_0^1 - \int_0^1 v\,du$$

$$= (x)\left(-\frac{1}{3}e^{-3x}\right)\Big|_0^1 + \int_0^1 \frac{1}{3}e^{-3x}\,dx$$

$$= -\frac{1}{3}xe^{-3x}\Big|_0^1 - \frac{1}{9}e^{-3x}\Big|_0^1$$

$$= \left(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right)\Big|_0^1$$

$$= -\frac{1}{3}e^{-3} - \frac{1}{9}e^{-3} - \left(0 - \frac{1}{9}\right) = \boxed{-\frac{4}{9}e^{-3} + \frac{1}{9}}$$

3. $\int x\cos(2x)\,dx$

I
L
$\boxed{A}$
T
E

$u = x, \quad dv = \cos(2x)\,dx$

$du = dx, \quad v = \frac{1}{2}\sin(2x)$

$$\int x\cos(2x)\,dx = uv - \int v\,du$$

$$= (x)\left(\frac{1}{2}\sin(2x)\right) - \int \frac{1}{2}\sin(2x)\,dx$$

$$= \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$$

what if: $\int 2x^2 e^x\,dx$

$u = 2x^2 \qquad dv = e^x\,dx$

$du = 4x\,dx \qquad v = e^x$

$$\int 2x^2 e^x\,dx = uv - \int v\,du$$

$$= 2x^2e^x - \int 4xe^x\,dx$$

$u = 4x \qquad dv = e^x\,dx$

$du = 4\,dx \qquad v = e^x$

$$= 2x^2e^x - \left(uv - \int v\,du\right)$$

$$= 2x^2e^x - \left(4xe^x - \int 4e^x\,dx\right)$$

$$= \boxed{2x^2e^x - 4xe^x + 4e^x + C}$$

4. $\int_0^{1/2} \arcsin x \, dx$

I
L
A
T
E

$u = \arcsin x \quad , \quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \quad , \quad v = x$

$$\int_0^{\frac{1}{2}} \arcsin x \, dx = uv \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} v \, du$$

$$= x \arcsin x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$$

u-sub

$u = 1 - x^2$

$du = -2x\,dx$

$$-\frac{1}{2} \int u^{-\frac{1}{2}} du$$

$$= -\cancel{\frac{1}{2}} \frac{u^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} = -\sqrt{1-x^2}$$

$$\left(x \arcsin x + \sqrt{1-x^2} \right) \Big|_0^{\frac{1}{2}}$$

$$\frac{1}{2} \arcsin \frac{1}{2} + \sqrt{1 - \frac{1}{4}} - \left(0 + \sqrt{1} \right)$$

$$\boxed{\frac{1}{2} \frac{\pi}{6} + \sqrt{\frac{3}{4}} - 1}$$

5. $\int e^{2x} \cos x \, dx$

$$u = e^{2x} \qquad dv = \cos x \, dx$$

$$du = 2e^{2x} \, dx \qquad v = \sin x$$

$$\int e^{2x} \cos x \, dx = uv - \int v \, du$$

$$= e^{2x} \sin x - \int 2e^{2x} \sin x \, dx$$

parts

$$u = 2e^{2x}, \quad dv = \sin x \, dx$$

$$du = 4e^{2x} \, dx, \quad v = -\cos x$$

$$= e^{2x} \sin x - \left(uv - \int v \, du \right)$$

$$= e^{2x} \sin x - \left(-2e^{2x} \cos x + \int 4e^{2x} \cos x \, dx \right)$$

$$\int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x - \int 4e^{2x} \cos x \, dx$$

add to other side

$$5 \int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x$$

$$\int e^{2x} \cos x \, dx = \frac{1}{5} \left(e^{2x} \sin x + 2e^{2x} \cos x \right) + C$$

Section 8.2

6. $\int \sin^2 x \cos^3 x \, dx$

$\int \sin^m x \cos^n x \, dx$

① if m or n (or both) is odd, factor one out of the odd expression.

power on cosine is odd, factor out $\cos x$.

$u = \sin x$

$du = \cos x \, dx$

$\cos^2 x + \sin^2 x = 1$

$$\int \underbrace{\sin^2 x}_{u^2} \, \underbrace{\cos^2 x}_{\downarrow} \, \underbrace{\cos x \, dx}_{du}$$

$1 - \sin^2 x$

$\downarrow$

$1 - u^2$

$$\int u^2 (1 - u^2) \, du$$

$$\int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

7. $\int_0^{\pi/8} \cos^2 4x\,dx$

$\int \sin^m x \cos^n x\,dx.$

all even powers, use

angle gets doubled

$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$$\int_0^{\frac{\pi}{8}} \frac{1}{2}(1+\cos(8x))\,dx = \frac{1}{2}\left[x + \frac{1}{8}\sin(8x)\right]\Bigg|_0^{\frac{\pi}{8}}$$

$$= \frac{1}{2}\left[\frac{\pi}{8} + \frac{1}{8}\cancelto{0}{\sin\pi} - (0)\right]$$

$$= \boxed{\frac{\pi}{16}}$$

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what if: $\int \cos^2\theta \sin^2\theta\,d\theta$

$$\int \frac{1}{2}(1+\cos 2\theta)\frac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$\frac{1}{4}\int (1-\cos^2 2\theta)\,d\theta = \frac{1}{4}\int \sin^2 2\theta\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right] + C$$

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx$$

$$= \int \left[\frac{1}{2}(1+\cos 2x)\right]^2 dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right)dx$$

$\vdots$

8. $\int \tan^5 x \sec^3 x \, dx$

$\int \tan^n x \sec^n x \, dx$

① even secant → factor out $\sec^2 x$

$u = \tan x$

② odd tangent → factor out $\sec x \tan x$

$u = \sec x$

- even secant and odd tangent do one of the above, not both!
- odd secant & even tangent break into sines & cosines

8. $\int \tan^5 x \sec^3 x \, dx$ odd tangent, factor out $\sec x \tan x$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$\int \underbrace{\tan^4 x}_{(\sec^2 x - 1)^2} \, \underbrace{\sec^2 x}_{u^2} \, \underbrace{\sec x \tan x \, dx}_{du}$$

$(\sec^2 x - 1)^2$
$\downarrow$
$(u^2 - 1)^2$

$\tan^2 x + 1 = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$\tan^4 x = (\sec^2 x - 1)^2$

$$\int (u^2 - 1)^2 u^2 \, du = \int (u^4 - 2u^2 + 1) u^2 \, du$$

$$= \int (u^6 - 2u^4 + u^2) \, du$$

$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C$$

$$= \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

9. $\int \frac{\sec^4 x}{\tan^7 x}\,dx$ even secant, factor out $\sec^2 x$

$\tan^2 x + 1 = u^2 + 1$

$$\int \frac{\sec^2 x}{\tan^7 x} \underbrace{\sec^2 x\,dx}_{du}$$

u^7

$u = \tan x$

$du = \sec^2 x\,dx$

$$\int \frac{u^2+1}{u^7}\,du = \int \left(u^{-5} + u^{-7}\right)du$$

$$= \frac{u^{-4}}{-4} + \frac{u^{-6}}{-6} + C = -\frac{1}{4\tan^4 x} - \frac{1}{6\tan^6 x} + C$$

10. $\displaystyle\int \frac{\sin^2(\ln x)}{x}\,dx$

u-sub

$u = \ln x$

$du = \dfrac{dx}{x}$

$$\int \sin^2 u \, du = \int \frac{1}{2}(1 - \cos 2u)\, du$$

$$= \frac{1}{2}\left(u - \frac{1}{2}\sin 2u\right) + C$$

$$= \frac{1}{2}\left(\ln x - \frac{1}{2}\sin(2\ln x)\right) + C$$

Section 8.3

11. $\int x^3\sqrt{4-x^2}\,dx =$

Look for:

① $a^2 - x^2 \longrightarrow x = a\sin\theta$

② $x^2 - a^2 \longrightarrow x = a\sec\theta$

③ $x^2 + a^2 \longrightarrow x = a\tan\theta$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$\int 8\sin^3\theta\sqrt{4-4\sin^2\theta}\;2\cos\theta\, d\theta$$

$$16\int \sin^3\theta\sqrt{4(\underbrace{1-\sin^2\theta}_{\cos^2\theta})}\;\cos\theta\, d\theta$$

odd sine, take one out

$$16\int \sin^3\theta\; 2\cos\theta\cos\theta\, d\theta = 32\int \sin^3\theta\cos^2\theta\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$= 32\int \underbrace{\sin^2\theta}_{1-\cos^2\theta\,\to\,1-u^2}\;\underbrace{\cos^2\theta}_{u^2}\;\underbrace{\sin\theta\, d\theta}_{-du}$$

$$= -32\int (1-u^2)u^2\, du$$

$$= -32\int (u^2-u^4)\, du$$

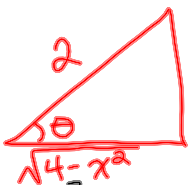
$$= -32\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right)$$

$$= -32\left(\frac{1}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta\right) + C$$

get back in terms of x.

in the first step, we have $x = 2\sin\theta$

$\frac{O}{H}$ $\frac{x}{2} = \sin\theta$



$\cos\theta = \frac{A}{H} = \frac{\sqrt{4-x^2}}{2}$

$$= -32\left(\frac{1}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 - \frac{1}{5}\left(\frac{\sqrt{4-x^2}}{2}\right)^5\right) + C$$

12. $\displaystyle\int_0^{2} \frac{x^3}{\sqrt{x^2+4}}\,dx =$

Form: $x^2 + a^2$ $\quad x = a\tan\theta$

$x = 2\tan\theta$ $\quad x=2 \rightarrow 2 = 2\tan\theta \rightarrow \theta = \frac{\pi}{4}$

$dx = 2\sec^2\theta\, d\theta$ $\quad x = 0 \rightarrow 0 = 2\tan\theta \rightarrow \theta = 0$

For $\tan\theta$: $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{4}} \frac{8\tan^3\theta}{\sqrt{\underbrace{4\tan^2\theta + 4}_{4(\tan^2\theta+1)}}}\, 2\sec^2\theta\, d\theta = 16\int_0^{\frac{\pi}{4}} \frac{\tan^3\theta\, \sec^2\theta}{2\sec\theta}\, d\theta$$

$4\sec^2\theta$

$$= 8\int_0^{\frac{\pi}{4}} \tan^3\theta\, \sec\theta\, d\theta$$

odd tangent factor out

$\sec\theta\tan\theta$

$u = \sec\theta$

$du = \sec\theta\tan\theta\, d\theta$

$$8\int_{\cancel{0}\to 1}^{\cancel{\frac{\pi}{4}}\to\sqrt{2}} \underbrace{\tan^2\theta}_{\sec^2\theta - 1 \;\to\; u^2-1}\, \underbrace{\sec\theta\tan\theta\, d\theta}_{du}$$

change limits

$u = \sec\theta$: $\theta = \frac{\pi}{4},\ u = \sqrt{2}$; $\theta = 0,\ u = 1$

$$8\int_1^{\sqrt{2}} (u^2 - 1)\, du$$

$$8\left(\frac{u^3}{3} - u\right)\Big|_1^{\sqrt{2}} = 8\left(\frac{(\sqrt{2})^3}{3} - \sqrt{2} - \left(\frac{1}{3} - 1\right)\right)$$

13. $\displaystyle\int \frac{1}{x^2\sqrt{16x^2-9}}\,dx =$ Kind of like $x^2 - a^2$ $\quad x = a\sec\theta$

$$\int \frac{dx}{x^2\sqrt{(4x)^2 - 9}}$$

$4x = 3\sec\theta$

$x = \frac{3}{4}\sec\theta$

$dx = \frac{3}{4}\sec\theta\tan\theta\,d\theta$

$$\int \frac{\frac{3}{4}\cancel{\sec\theta}\tan\theta\,d\theta}{\frac{9}{16}\sec^{\cancel{2}}\theta\sqrt{\underbrace{9\sec^2\theta - 9}_{\substack{9(\sec^2\theta-1)\\ 9\tan^2\theta}}}} \to \frac{\cancel{3}}{\cancel{4}}\cdot\frac{\cancel{16}^{\,4}}{9}\int\frac{\cancel{\tan\theta}\,d\theta}{\sec\theta\cdot\cancel{3}\cancel{\tan\theta}}$$

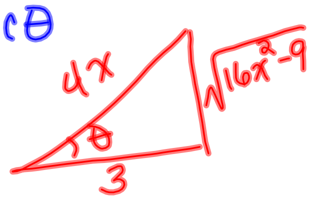
$$= \frac{4}{9}\int\frac{d\theta}{\sec\theta}$$

$$= \frac{4}{9}\int\cos\theta\,d\theta$$

$$= \frac{4}{9}\sin\theta + C$$

$4x = 3\sec\theta$

$\frac{4x}{3} = \sec\theta \quad \frac{H}{A}$



$$= \boxed{\frac{4}{9}\cdot\frac{\sqrt{16x^2-9}}{4x} + C}$$

14. $\int \frac{dx}{\sqrt{x^2+4x+8}}$ [redacted] $=$ $\underbrace{x^2+4x+4}_{(x+2)^2} + 8 - 4 \quad (x+2)^2+4$ $\qquad \left(\frac{4}{2}\right)^2$

$$\int \frac{dx}{\sqrt{(x+2)^2+4}}$$

$x+2 = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

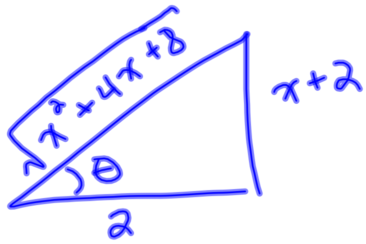
$$\int \frac{2\sec^2\theta \, d\theta}{\sqrt{\underbrace{4\tan^2\theta+4}_{\substack{4(\tan^2\theta+1) \\ 4\sec^2\theta}}}} = \int \frac{2\sec^2\theta \, d\theta}{2\sec\theta}$$

$$= \int \sec\theta \, d\theta$$

$x+2 = 2\tan\theta$

$\frac{x+2}{2} = \tan\theta$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$



$$= \ln\left|\frac{\sqrt{x^2+4x+8}}{2} + \frac{x+2}{2}\right| + C$$

15. $\int \sqrt{1-4x^2}\,dx = \int \sqrt{1-(2x)^2}\,dx$

$$2x = \sin\theta$$

$$2\,dx = \cos\theta\,d\theta$$

$$dx = \frac{\cos\theta\,d\theta}{2}$$

$$\int \sqrt{\underbrace{1-\sin^2\theta}_{\cos^2\theta}}\ \frac{\cos\theta}{2} = \frac{1}{2}\int \cos^2\theta\,d\theta$$

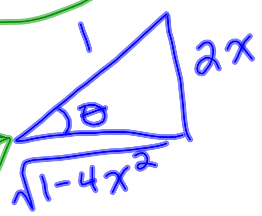
$$= \frac{1}{2}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$2x = \sin\theta$

$\theta = \arcsin(2x)$

$\cancel{2}\sin\theta\cos\theta$

$$= \frac{1}{4}\left(\theta + \frac{1}{\cancel{2}}\sin 2\theta\right) + C$$



$$= \frac{1}{4}\left(\arcsin(2x) + (2x)\left(\sqrt{1-4x^2}\right)\right) + C$$