

Section 8.4

$$1. \int \frac{x^3 + 2x + 1}{x(x+4)} dx$$

$$\int \frac{\text{polynomial}}{\text{polynomial}}$$

① is degree on bottom bigger than degree on top? If so, proceed. If not, do long division

Since power on bottom is degree 2 & power on top is degree 3, must first long divide:

$$\begin{array}{r} x-4 \leftarrow Q \\ x^2+4x \overline{) x^3+2x+1} \\ \underline{-x^3+4x^2} \\ -4x^2+2x+1 \\ \underline{+4x^2+16x} \\ 18x+1 \leftarrow R \end{array}$$

$$Q + \frac{R}{D}$$

$$\int \left(x-4 + \frac{18x+1}{x^2+4x} \right) dx$$

$$= \frac{x^2}{2} - 4x + \text{PFD}$$

$$\cancel{x(x+4)} \left(\frac{18x+1}{\cancel{x(x+4)}} \right) = \left(\frac{A}{x} + \frac{B}{x+4} \right) x(x+4)$$

$$18x+1 = A(\underline{x+4}) + B\underline{x}$$

$$x=0:$$

$$1 = A(4) \rightarrow \boxed{A = \frac{1}{4}}$$

$$x=-4:$$

$$-71 = B(-4) \rightarrow \boxed{B = \frac{71}{4}}$$

$$\left(x-4 + \frac{18x+1}{x^2+4x} \right) dx = \int \left(x-4 + \frac{1}{4x} + \frac{71}{4(x+4)} \right) dx$$

u-sub u = x+4

$$= \left[\frac{x^2}{2} - 4x + \frac{1}{4} \ln|x| + \frac{71}{4} \ln|x+4| + C \right]$$

$$2. \int_1^2 \frac{dx}{x(x^2 + 2x + 1)} = \int_1^2 \frac{dx}{x(x+1)^2}$$

$$\cancel{x(x+1)^2} \left(\frac{1}{\cancel{x(x+1)^2}} \right) = \left(\frac{\cancel{A}^1}{x} + \frac{\cancel{B}^{-1}}{x+1} + \frac{\cancel{C}^{-1}}{(x+1)^2} \right) x(x+1)^2$$

$$1 = \cancel{A}^1 (x+1)^2 + Bx(x+1) + \cancel{C}^{-1} x$$

$$x=0: \boxed{1=A}$$

$$x=-1: 1 = C(-1) \rightarrow \boxed{C=-1}$$

$$\rightarrow 1 = (x+1)^2 + Bx(x+1) - x$$

$$\text{choose } x=1: 1 = 4 + 2B - 1 \rightarrow 1 = 3 + 2B$$

$$-2 = 2B$$

$$\boxed{B=-1}$$

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx$$

$$u = x+1$$

$$\int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{x+1}$$

$$\left(\ln|x| - \ln|x+1| + \frac{1}{x+1} \right) \Big|_1^2$$

$$\left(\ln \left| \frac{x}{x+1} \right| + \frac{1}{x+1} \right) \Big|_1^2 = \ln \frac{2}{3} + \frac{1}{3} - \left(\ln \frac{1}{2} + \frac{1}{2} \right)$$

$$3. \int \frac{3x^2 - 4x + 5}{(x-1)(x^2+1)} dx$$

$$\frac{3x^2 - 4x + 5}{\cancel{(x-1)}(x+1)} = \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+1} \right) (x-1)(x^2+1)$$

$$3x^2 - 4x + 5 = \overset{2}{A}(x^2+1) + (Bx+C)(\underline{x-1})$$

$$x=1: \quad 4 = A(2) \rightarrow \boxed{A=2}$$

$$3x^2 - 4x + 5 = 2x^2 + 2 + Bx^2 - Bx + Cx - C$$

$$\underline{3x^2 - 4x + 5} = \underline{(2+B)}x^2 + (C-B)x + \underline{2-C}$$

$$3 = 2+B \rightarrow \boxed{B=1}$$

$$5 = 2-C \rightarrow \boxed{C=-3}$$

check:

$$-4 \stackrel{?}{=} C-B$$

$$-4 = -3-1 \text{ yes}$$

$$\int \left(\frac{2}{x-1} + \frac{x-3}{x^2+1} \right) dx = \int \left(\frac{2}{x-1} + \frac{x}{x^2+1} - \frac{3}{x^2+1} \right) dx$$

split up!

u-sub
 $u = x^2+1$
 $du = 2x dx$
 $\frac{1}{2} \int \frac{du}{u}$

recall

$$\int \frac{dx}{x^2+1} = \arctan x + C$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\frac{1}{2} \ln|u|$$

$$\frac{1}{2} \ln|x^2+1|$$

$$= 2 \ln|x-1| + \frac{1}{2} \ln|x^2+1| - 3 \arctan x + C$$

$$4. \int \frac{x+6}{(x^2+1)(x^2+4)} dx$$

$$\frac{x+6}{\cancel{(x^2+1)}\cancel{(x^2+4)}} = \left(\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \right) (x^2+1)(x^2+4)$$

$$x+6 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$x+6 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

$$0x^3 + 0x^2 + x + 6 = (A+C)x^3 + (B+D)x^2 + (4A+C)x + 4B+D$$

$$\begin{cases} 0 = A+C \\ 0 = B+D \\ 1 = 4A+C \\ 6 = 4B+D \end{cases}$$

$$\begin{aligned} 0 = A+C &\rightarrow A = -C \\ 1 = 4A+C &\rightarrow 1 = -4C+C \end{aligned}$$

$$1 = -3C$$

$$\boxed{C = -\frac{1}{3}} \\ \boxed{A = \frac{1}{3}}$$

$$4. \int \frac{x+6}{(x^2+1)(x^2+4)} dx$$

$$\int \left(\frac{\frac{1}{3}x+2}{x^2+1} + \frac{-\frac{1}{3}x-2}{x^2+4} \right) dx$$

$$0 = B+D \rightarrow B = -D$$

$$6 = 4B+D \rightarrow 6 = -4D+D$$

$$6 = -3D$$

$$\boxed{-2 = D} \\ \boxed{B = 2}$$

$$\int \left(\underbrace{\frac{\frac{1}{3}x}{x^2+1}}_{u\text{-sub}} + \underbrace{\frac{2}{x^2+1}}_{\substack{\arctan \\ a=1}} + \underbrace{\frac{-\frac{1}{3}x}{x^2+4}}_{u\text{-sub}} - \underbrace{\frac{2}{x^2+4}}_{\substack{\arctan \\ a=2}} \right) dx$$

$$\frac{1}{3} \cdot \frac{1}{2} \ln|x^2+1| + 2 \arctan x + -\frac{1}{3} \cdot \frac{1}{2} \ln|x^2+4| - 2 \cdot \frac{1}{2} \arctan \frac{x}{2} + C$$

5. Determine whether the following improper integrals converge or diverge. If it converges, find the value of the integral. If it diverges, explain why.

a.) $\int_2^{\infty} \frac{x}{x^2+1} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{x}{x^2+1} dx$

$$u = x^2 + 1$$
$$du = 2x dx$$

$$\frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln u$$
$$= \frac{1}{2} \ln(x^2 + 1)$$

Fact: $\ln \infty = \infty!$

$$= \lim_{t \rightarrow \infty} \left. \frac{1}{2} \ln(x^2 + 1) \right|_2^t$$

$$= \lim_{t \rightarrow \infty} \left(\frac{1}{2} \ln(t^2 + 1) - \frac{1}{2} \ln 5 \right)$$

$$= \boxed{\infty \rightarrow \text{integral diverges}}$$

$$b.) \int_e^{\infty} \frac{1}{x(\ln x)^4} dx = \lim_{t \rightarrow \infty} \int_e^t \frac{1}{x(\ln x)^4} dx$$

$$u = \ln x$$

$$du = \frac{dx}{x}$$

$$= \lim_{t \rightarrow \infty} \left[-\frac{1}{3(\ln x)^3} \right] \Big|_e^t$$

$$\int \frac{du}{u^4} = -\frac{1}{3u^3}$$

note: $\ln e = 1$

$$= -\frac{1}{3(\ln x)^3}$$

$$= \lim_{t \rightarrow \infty} \left[-\frac{1}{3(\ln t)^3} + \frac{1}{3} \right] = \boxed{\frac{1}{3} \text{ integral converges}}$$

$$c.) \int_{-\infty}^0 x e^x dx = \lim_{t \rightarrow -\infty} \int_t^0 x e^x dx$$

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parts: $u = x$ $dv = e^x dx$
 $du = dx$ $v = e^x$

$$\begin{aligned} \int x e^x dx &= uv - \int v du \\ &= x e^x - \int e^x dx \\ &= x e^x - e^x \end{aligned}$$

$$\lim_{t \rightarrow -\infty} (x e^x - e^x) \Big|_t^0$$

$$\lim_{t \rightarrow -\infty} \left[0 - 1 - \left(\overset{\textcircled{2}}{\underbrace{t e^t}_{\downarrow 0}} - \overset{\textcircled{1}}{\underbrace{e^t}_{\downarrow 0}} \right) \right]$$

$$\textcircled{1} \lim_{t \rightarrow -\infty} \frac{t}{e^t} = \frac{-\infty}{e^\infty} = \frac{1}{e^\infty} = 0$$

$= -1$ converges

$$\textcircled{2} \lim_{t \rightarrow -\infty} t e^t$$

$$\lim_{t \rightarrow -\infty} \frac{t}{e^{-t}} \quad \frac{\infty}{\infty}$$

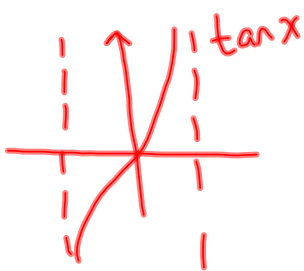
$$\stackrel{L}{=} \lim_{t \rightarrow -\infty} \frac{1}{-e^{-t}}$$

$$= \lim_{t \rightarrow -\infty} -e^t = 0$$

$$d.) \int_{-\infty}^{\infty} \frac{dx}{x^2+9} = \textcircled{1} \int_{-\infty}^0 \frac{dx}{x^2+9} + \textcircled{2} \int_0^{\infty} \frac{dx}{x^2+9}$$

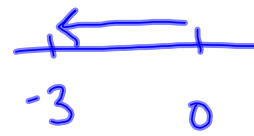
$$\begin{aligned} \textcircled{1} \lim_{t \rightarrow -\infty} \int_t^0 \frac{dx}{x^2+9} &= \lim_{t \rightarrow -\infty} \frac{1}{3} \arctan \frac{x}{3} \Big|_t^0 \\ &= \lim_{t \rightarrow -\infty} \frac{1}{3} \left[\underset{\substack{\downarrow \\ 0}}{\arctan 0} - \underset{\substack{\downarrow \\ -\frac{\pi}{2}}}{\arctan \frac{t}{3}} \right] \\ &= \frac{1}{3} \left(\frac{\pi}{2} \right) = \boxed{\frac{\pi}{6}} \leftarrow \begin{array}{l} \text{finite,} \\ \text{move to} \\ \text{second integral} \end{array} \end{aligned}$$

$$\textcircled{2} \lim_{t \rightarrow \infty} \int_0^t \frac{dx}{x^2+9} = \lim_{t \rightarrow \infty} \frac{1}{3} \arctan \frac{x}{3} \Big|_0^t$$



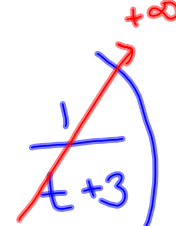
$$\begin{aligned} &= \lim_{t \rightarrow \infty} \frac{1}{3} \left(\underset{\substack{\downarrow \\ \frac{\pi}{2}}}{\arctan \frac{t}{3}} - \underset{\substack{\downarrow \\ 0}}{\arctan 0} \right) \\ &= \boxed{\frac{\pi}{6}} \end{aligned}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^2+9} = \frac{\pi}{6} + \frac{\pi}{6}}$$

$$e.) \int_{-3}^0 \frac{dx}{(x+3)^2} = \lim_{t \rightarrow -3^+} \int_t^0 \frac{dx}{(x+3)^2}$$


$$u = x+3$$

$$\int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{x+3}$$

$$= \lim_{t \rightarrow -3^+} \left[-\frac{1}{x+3} \right]_t^0 = \lim_{t \rightarrow -3^+} \left(-\frac{1}{3} + \frac{1}{t+3} \right)$$


$$= \boxed{\infty \leftarrow \text{diverges}}$$

$$f.) \int_{-1}^{32} \frac{1}{\sqrt[5]{x}} dx = \textcircled{1} \int_{-1}^0 x^{-\frac{1}{5}} dx + \textcircled{2} \int_0^{32} x^{-\frac{1}{5}} dx$$

$$\textcircled{1} \lim_{t \rightarrow 0^-} \int_{-1}^t x^{-\frac{1}{5}} dx = \lim_{t \rightarrow 0^-} \left. \frac{5}{4} x^{\frac{4}{5}} \right|_{-1}^t$$

$$= \lim_{t \rightarrow 0^-} \frac{5}{4} \left[\cancel{t^{\frac{4}{5}}} - (-1)^{\frac{4}{5}} \right]$$

$$\textcircled{2} \lim_{t \rightarrow 0^+} \int_t^{32} x^{-\frac{1}{5}} dx$$

$$\lim_{t \rightarrow 0^+} \left. \frac{5}{4} x^{\frac{4}{5}} \right|_t^{32} = \lim_{t \rightarrow 0^+} \frac{5}{4} \left(\cancel{32^{\frac{4}{5}}} - \cancel{t^{\frac{4}{5}}} \right)$$

$$= \frac{5}{4} (16) = \boxed{20}$$

ADD

$$\boxed{\int_{-1}^{32} x^{-\frac{1}{5}} dx = 20 - \frac{5}{4}}$$

6. Determine whether the following integrals converge or diverge using the comparison theorem:

a.) $\int_0^{\infty} \frac{1}{x^{10} + e^{5x}} dx$ compare $\frac{1}{x^{10} + e^{5x}}$ with $\frac{1}{e^{5x}}$

$$0 \leq \int_0^{\infty} \frac{dx}{x^{10} + e^{5x}} \leq \int_0^{\infty} \frac{dx}{e^{5x}} = \lim_{t \rightarrow \infty} \int_0^t e^{-5x} dx$$

Larger integral converges, so does smaller.
 note: Larger integral diverges, means nothing!

$$= \lim_{t \rightarrow \infty} -\frac{1}{5} e^{-5x} \Big|_0^t$$

$$= \lim_{t \rightarrow \infty} -\frac{1}{5} [e^{-5t} - 1]$$

$$= \boxed{\frac{1}{5}}$$

$$b.) \int_2^{\infty} \frac{x}{x^{3/2} - x} dx$$

$$\int_2^{\infty} \frac{x}{x^{\frac{3}{2}} - x} dx \geq \int_2^{\infty} \frac{x}{x^{\frac{3}{2}}} dx$$

$$= \lim_{t \rightarrow \infty} \int_2^t x^{-\frac{1}{2}} dx$$

$$= \lim_{t \rightarrow \infty} \left. \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right|_2^t$$

$$= \lim_{t \rightarrow \infty} 2\sqrt{x} \Big|_2^t$$

$$= \lim_{t \rightarrow \infty} (2\sqrt{t} - 2\sqrt{2})$$

smaller integral
diverges, so does larger.

$$c.) \int_1^{\infty} \frac{\cos^2 x}{x^4} dx \leq \int_1^{\infty} \frac{1}{x^4} dx \quad \text{since } \cos x \leq 1$$

$$= -\frac{1}{3x^3} \Big|_1^{\infty}$$

$$= -\frac{1}{3(\infty)^3} + \frac{1}{3} = \frac{1}{3}$$

larger
converges,
so does smaller!

Section 9.3

7. Find the length of the curve $y = 2x^{3/2}$, $0 \leq x \leq \frac{1}{4}$. ①

if equation involves x & y

$$\textcircled{1} \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\textcircled{2} \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$L = \int_0^{\frac{1}{4}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$y = 2x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = 3x^{\frac{1}{2}} = 3\sqrt{x}$$

$$L = \int_0^{\frac{1}{4}} \sqrt{1 + 9x} dx \quad \begin{array}{l} u\text{-sub, } u = 1 + 9x \\ du = 9 dx \end{array} \quad \begin{array}{l} x = \frac{1}{4}, u = \frac{13}{4} \\ x = 0, u = 1 \end{array}$$

$$L = \int_1^{\frac{13}{4}} \frac{1}{9} \sqrt{u} du$$

$$= \frac{1}{9} \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{\frac{13}{4}}$$

$$= \frac{2}{27} \left(\left(\frac{13}{4}\right)^{\frac{3}{2}} - 1 \right)$$

8. Find the length of the curve $x = y^2 - \frac{\ln(y)}{8}$ from $y = 1$ to $y = e$.

(2)

$$L = \int_1^e \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$x = y^2 - \frac{\ln y}{8}$$

$$\frac{dx}{dy} = 2y - \frac{1}{8y}$$

$$L = \int_1^e \sqrt{1 + 4y^2 - \frac{1}{2} + \frac{1}{64y^2}} dy$$

$$\left(\frac{dx}{dy}\right)^2 = \left(2y - \frac{1}{8y}\right)\left(2y - \frac{1}{8y}\right)$$

$$= \int_1^e \sqrt{4y^2 + \frac{1}{2} + \frac{1}{64y^2}} dy$$

$$= 4y^2 - \frac{1}{4} - \frac{1}{4} + \frac{1}{64y^2}$$

$$= 4y^2 - \frac{1}{2} + \frac{1}{64y^2}$$

$$= \int_1^e \sqrt{\left(2y + \frac{1}{8y}\right)^2} dy$$

$$= \int_1^e \left(2y + \frac{1}{8y}\right) dy = \left(y^2 + \frac{1}{8} \ln y\right) \Big|_1^e$$

$$= e^2 + \frac{1}{8} - (1 + 0)$$

9. Find the length of the parametric curve $x = 3t - t^3$,
 $y = 3t^2$, $0 \leq t \leq 2$.

Parametric $L = \int \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt$

$$L = \int_0^2 \sqrt{(6t)^2 + (3 - 3t^2)^2} dt$$

$$= \int_0^2 \sqrt{\underline{36t^2} + 9 - \underline{18t^2} + 9t^4} dt$$

$$= \int_0^2 \sqrt{9t^4 + 18t^2 + 9} dt$$

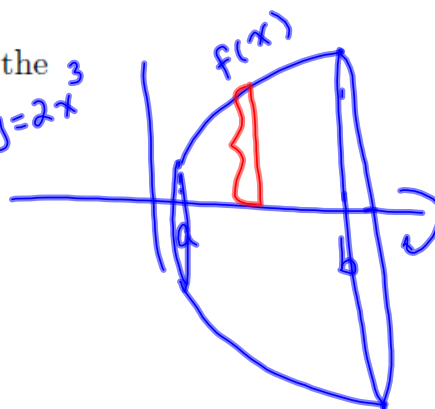
$$= \int_0^2 \sqrt{(3t^2 + 3)^2} dt$$

$$= \int_0^2 (3t^2 + 3) dt = (t^3 + 3t) \Big|_0^2 = 8 + 6 = 14$$

Section 9.4

10. Find the surface area obtained by revolving the ³ given curve about the indicated axis.

a.) $y = 2x^3$, $0 \leq x \leq 1$ about the x axis.



$$SA = \int 2\pi r \cdot \text{arc length} \frac{dx}{dt}$$

$$SA = \int_0^1 2\pi \cdot 2x^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= 4\pi \int_0^1 x^3 \sqrt{1 + (6x^2)^2} dx$$

$$= 4\pi \int_0^1 x^3 \sqrt{1 + 36x^4} dx$$

$$= 4\pi \int_1^{37} \frac{1}{144} \sqrt{u} du$$

$$= \frac{4\pi}{144} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{37}$$

$r = \text{opposite of axis of revolution}$

@ x -axis: $r = y$

@ y -axis: $r = x$

u -sub

$u = 1 + 36x^4$

$du = 144x^3 dx$

$\begin{matrix} 2 \\ 36 \\ 4 \\ 144 \end{matrix}$

b.) $y^2 = x + 2$, $1 \leq y \leq 3$ about the x axis. $\rightarrow r = y$

\downarrow
 dy

$$\rightarrow x = y^2 - 2$$

$$\frac{dx}{dy} = 2y$$

$$SA = \int_1^3 2\pi r \text{arclength} dy$$

$$= \int_1^3 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$= \int_1^3 2\pi y \sqrt{1 + 4y^2} dy$$

$$= \frac{2\pi}{8} \int_5^{37} u^{\frac{1}{2}} du$$

$$= \frac{\pi}{4} \frac{2}{3} u^{\frac{3}{2}} \Big|_5^{37}$$

u-sub
 $u = 1 + 4y^2$
 $du = 8y dy$

c.) $y = x^2 + 1$, $0 \leq x \leq 1$, about the y axis. $\rightarrow r = x$
 \downarrow
 dx

$$A = \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$y = x^2 + 1$
 $\frac{dy}{dx} = 2x$

$$= \int_0^1 2\pi x \sqrt{1 + 4x^2} dx$$

$$= \frac{2\pi}{8} \int_1^5 u^{\frac{1}{2}} du$$

$$= \frac{\pi}{4} \frac{2}{3} u^{\frac{3}{2}} \Big|_1^5$$

u -sub
 $u = 1 + 4x^2$
 $du = 8x dx$

d.) $y = \sqrt{4x}$, $0 \leq x \leq 1$, about the x axis. $\rightarrow r = y$

$$y^2 = 4x$$

$$x = \frac{1}{4}y^2$$

$$0 \leq y \leq 2$$

$$\frac{dx}{dy} = \frac{1}{2}y$$

$$A = \int_0^2 2\pi y \sqrt{1 + \frac{1}{4}y^2} dy$$

$$u = 1 + \frac{1}{4}y^2$$
$$\vdots$$

e.) $x = \ln(3y + 1)$, $0 \leq y \leq 2$, about the y axis, then the x axis. Set up the integral that gives the surface area. Do not integrate.

$$x = \ln(3y + 1)$$
$$\frac{dx}{dy} = \frac{3}{3y + 1}$$

@ y -axis: $r = x = \ln(3y + 1)$

$$A = \int_0^2 2\pi \ln(3y + 1) \sqrt{1 + \left(\frac{3}{3y + 1}\right)^2} dy$$

@ x -axis: $r = y$.

$$A = \int_0^2 2\pi y \sqrt{1 + \left(\frac{3}{3y + 1}\right)^2} dy$$

f.) $x = \sin(3t)$, $y = \cos(3t)$, $0 \leq t \leq \frac{\pi}{12}$. about the y axis. $r = x = \sin(3t)$

$$A = \int_0^{\frac{\pi}{12}} 2\pi \sin(3t) \sqrt{(-3\sin(3t))^2 + (3\cos(3t))^2} dt$$

$$= \int_0^{\frac{\pi}{12}} 2\pi \sin(3t) \sqrt{9(\underbrace{\sin^2 3t + \cos^2 3t}_{\downarrow 1})} dt$$

$$= 6\pi \int_0^{\frac{\pi}{12}} \sin(3t) dt = -6\pi \frac{1}{3} \cos(3t) \Big|_0^{\frac{\pi}{12}}$$

$$= -2\pi \left(\frac{\sqrt{2}}{2} - 1 \right)$$