Section 10.1

1. Find the fourth term of the sequence $\{\frac{n}{n+1}\}_{n=2}^{\infty}$

$$\left\{\frac{n}{n+1}\right\}_{n=2}^{\infty} = \left\{\underset{a_2}{\frac{2}{3}}, \underset{a_3}{\frac{3}{4}}, \underset{a_4}{\frac{4}{5}}, \underset{a_5}{\frac{5}{6}}, \ldots\right\}$$

4^{th} term is $a_5 = \frac{5}{6}$

$$\boxed{\lim_{n\to\infty} \frac{n}{n+1} = 1}$$

2. Find a general formula for the sequence
$\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots$

$$\left\{\frac{1}{2\cdot 2}, \frac{1}{3\cdot 2}, \frac{1}{4\cdot 2}, \frac{1}{5\cdot 2}, \ldots\right\} = \left\{\frac{1}{2n}\right\}_{n=2}^{\infty}$$

$$\lim_{n\to\infty} \frac{1}{2n} = 0$$

3. Determine whether the following sequences converge or diverge. If the sequence converges, find the limit. If the sequence diverges, explain why.

a.) $a_n = \dfrac{n^3}{n^2 + 500n - 2}$

$$\lim_{n\to\infty} \frac{n^3}{n^2 + 500n - 2} = \infty \quad \text{diverges}$$

b.) $a_n = \ln(2n+1) - \ln(5n+4) = \ln\left(\dfrac{2n+1}{5n+4}\right)$

$$\lim_{n\to\infty} \ln\left(\overset{\to \frac{2}{5}}{\frac{2n+1}{5n+4}}\right) = \boxed{\ln \frac{2}{5}}$$

c.) $a_n = \frac{5\cos n}{n}$

$$-1 \le \cos n \le 1$$

$$-5 \le 5\cos n \le 5$$

as $n \to \infty$,

$$\underbrace{\frac{-5}{n}}_{\to 0} \le \frac{5\cos n}{n} \le \underbrace{\frac{5}{n}}_{\to 0}$$

$$\frac{5\cos n}{n} \to 0$$

$5\cos n$ is bounded by ± 5

bottom is growing without bound,

so $\lim_{n\to\infty} \frac{5\cos n}{n} = 0$

d.) $a_n = \frac{(-1)^n}{n} + 6$

$$\lim_{n\to\infty} \left(\frac{(-1)^n}{n} + 6 \right) = \lim_{n\to\infty} \frac{(-1)^n}{n} + \lim_{n\to\infty} (6)$$

$$\lim_{n\to\infty} \frac{(-1)^n + 6n}{n} = \lim_{n\to\infty} \left(\underbrace{\frac{(-1)^n}{n}}_{\to 0} + \underbrace{\frac{6n}{n}}_{\to 6} \right)$$

$$= 0 + 6$$

$$= \boxed{6}$$

e.) $a_n = \frac{(-1)^n n}{5n+6}$

$$\lim_{n\to\infty} \frac{(-1)^n n}{5n+6} \to \frac{1}{5}$$

$\longrightarrow$ n even $\to \frac{1}{5}$

n odd $\to -\frac{1}{5}$

diverges by oscillation

4. Prove the sequence $a_n = \frac{n}{n^2+1}$ is bounded.

$$\left\{ \frac{n}{n^2+1} \right\}_{n=1}^{\infty} = \left\{ \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \dots \right\} \qquad \lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

$$0 < a_n \leq \frac{1}{2}$$

5. Prove the sequence $a_n = \frac{1}{n^3}$ is a decreasing sequence.

compare a_n and a_{n+1}

if $a_n < a_{n+1}$ increasing

* if $a_{n+1} < a_n$ decreasing

$$a_n = \frac{1}{n^3}, \quad a_{n+1} = \frac{1}{(n+1)^3}$$

$$\frac{1}{(n+1)^3} < \frac{1}{n^3} \implies a_{n+1} < a_n \text{ decreasing}$$

6. Prove the sequence $a_n = \frac{n}{\ln n}$ is an increasing sequence.

$$a_n = \frac{n}{\ln n}, \quad a_{n+1} = \frac{n+1}{\ln(n+1)}$$

not easily comparable

recall: $f' > 0 \to f$ increasing

$f' < 0 \to f$ decreasing

$$\left(\frac{n}{\ln n}\right)' = \frac{(1)(\ln n) - (n)\left(\frac{1}{n}\right)}{(\ln n)^2} = \frac{\ln n - 1}{(\ln n)^2} > 0 \text{ for } n > 2$$

$\therefore$ increasing sequence

7. For the recursive sequence given, find the 3rd term and find the value of the limit.

$a_1 = 2$, $a_{n+1} = 2 + \frac{1}{4}a_n$.

$$a_{n+1} = f(a_n)$$

$$a_1 = 2$$

$$a_2 = 2 + \frac{1}{4}a_1 = 2 + \frac{1}{4}(2) = 2 + \frac{1}{2} = \frac{5}{2} = 2.5$$

$$a_3 = 2 + \frac{1}{4}a_2 = 2 + \frac{1}{4}\cdot\frac{5}{2} = 2 + \frac{5}{8} = \frac{21}{8} = 2.625$$

given $\{a_n\}$ converges, find the limit.

$$a_{n+1} = 2 + \frac{1}{4}a_n$$

suppose $\lim_{n\to\infty} a_n = L \rightarrow a_\infty = L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(2 + \frac{1}{4}a_n\right)$$

$$\cancelto{L}{a_\infty} = 2 + \frac{1}{4}\cancelto{L}{a_\infty}$$

$$L = 2 + \frac{1}{4}L \rightarrow \frac{3}{4}L = 2$$

$$L = \frac{8}{3}$$

① $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + \dots$

② n^{th} partial sum $\quad S_n = a_1 + a_2 + \dots + a_n$

③ $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

④ Test for divergence $\quad \lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

if $\lim_{n\to\infty} a_n = 0$, the test fails

Section 10.2

8. Use the Test For Divergence to show the series diverges:

$$\sum_{n=1}^{\infty} \frac{n^2}{3(n+1)(n+2)}$$

T.D. $\lim_{n\to\infty} \frac{n^2}{3(n+1)(n+2)} = \frac{1}{3} \neq 0$

series diverges.

9. Explain why the Test for Divergence is inconclusive when applied to the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$.

$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0 \rightarrow$ test fails.

$\frac{1}{n} \to 0$

do another test learned in section 10.3

10. If the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is $s_n = \frac{n+1}{n+4}$, find:

a.) s_{100}, that is $\sum_{n=1}^{100} a_n = ?$

$S_n = \frac{n+1}{n+4}$ = sum of first n terms.

$S_{100} = \frac{101}{104}$ = sum of first 100 terms

b.) The sum of the series, that is $\sum_{n=1}^{\infty} a_n = ?$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n+4} = 1$$

c.) A general formula for a_n, then find a_6.

$a_n = S_n - S_{n-1}$ $\quad S_n = \frac{n+1}{n+4}$

$$a_n = \frac{n+1}{n+4} - \frac{n}{n+3}$$

$$= \frac{(n+1)(n+3) - n(n+4)}{(n+4)(n+3)} = \frac{n^2+4n+3-n^2-4n}{(n+4)(n+3)} = \boxed{\frac{3}{(n+4)(n+3)} = a_n}$$

$\lim_{n\to\infty} a_n = 0$

$$a_6 = \frac{3}{(10)(9)} = \frac{3}{90}$$

Three ways to find sum

① If S_n is known, then
$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

② If $\sum_{n=1}^{\infty} a_n$ is telescoping

① Find S_n
② $\lim_{n\to\infty} S_n$

③ $\sum_{n=1}^{\infty} a_n$ is a geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

if $|r|<1$.

if $|r|>1$, series diverges

11. Find the sum of the series:

$\cancel{\sum_{n=1}^{\infty}\left(\sin\frac{1}{n}-\sin\frac{1}{n+1}\right)}$

a.) $\displaystyle\sum_{n=2}^{\infty}\left(\cos\left(\frac{1}{n+3}\right)-\cos\left(\frac{1}{n+4}\right)\right)$

T.D. $\displaystyle\lim_{n\to\infty}\left(\cos\frac{1}{n+3}-\cos\frac{1}{n+4}\right)=\cos(0)-\cos(0)=1-1=0$ Fails

step1 Find $S_n = a_2 + a_3 + a_4 + \dots + a_n$

$$S_n = \underbrace{\cos\left(\tfrac{1}{5}\right)-\cos\left(\cancel{\tfrac{1}{6}}\right)}_{a_2} + \underbrace{\cancel{\cos\tfrac{1}{6}} - \cancel{\cos\tfrac{1}{7}}}_{a_3} + \underbrace{\cancel{\cos\tfrac{1}{7}} - \cancel{\cos\tfrac{1}{8}}}_{a_4} + \cancel{+}\dots$$

$$+\dots \underbrace{\cancel{\cos\tfrac{1}{n+3}} - \cos\tfrac{1}{n+4}}_{a_n}$$

$$\boxed{S_n = \cos\left(\tfrac{1}{5}\right) - \cos\left(\tfrac{1}{n+4}\right)}$$

$$\text{sum of series} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\cos\frac{1}{5}-\cos\frac{1}{n+4}\right)$$

$$= \boxed{\cos\frac{1}{5}-1}$$ ← sum!

b.) $\cancel{\sum_{n=1}^{\infty}\frac{1}{n(n+2)}}$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+3)}$

PFD: $\displaystyle\frac{1}{(n+1)(n+3)} = \frac{A}{n+1}+\frac{B}{n+3}$ ($A \to \frac{1}{2}$, $B \to -\frac{1}{2}$)

$1 = A(n+3) + B(n+1)$

$n=-1$: $1 = A(2) \longrightarrow A=\frac{1}{2}$

$n=-3$: $1 = B(-2) \longrightarrow B=-\frac{1}{2}$

$$\sum_{n=1}^{\infty}\left(\frac{\frac{1}{2}}{n+1}-\frac{\frac{1}{2}}{n+3}\right)$$

$S_n = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$

$$S_n = \underbrace{\frac{\frac12}{2}-\cancel{\frac{\frac12}{4}}}_{a_1} + \underbrace{\frac{\frac12}{3}-\cancel{\frac{\frac12}{5}}}_{a_2} + \underbrace{\cancel{\frac{\frac12}{4}}-\frac{\frac12}{6}}_{a_3} + \underbrace{\cancel{\frac{\frac12}{5}}-\cancel{\frac{\frac12}{7}}}_{a_4} + \dots + \underbrace{\cancel{\frac{\frac12}{n}}-\frac{\frac12}{n+2}}_{a_{n-1}} + \underbrace{\cancel{\frac{\frac12}{n+1}}-\frac{\frac12}{n+3}}_{a_n}$$

$$S_n = \frac{\frac12}{2}+\frac{\frac12}{3}-\frac{\frac12}{n+2}-\frac{\frac12}{n+3}$$

$$\text{sum of series} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{\frac12}{2}+\frac{\frac12}{3}-\cancel{\frac{\frac12}{n+2}}^{\,0}-\cancel{\frac{\frac12}{n+3}}^{\,0}\right]$$

$$= \boxed{\frac{1}{4}+\frac{1}{6}}$$

added ex: $\displaystyle\sum_{n=2}^{\infty}\ln\frac{n}{5n+2}$ T.D. $\displaystyle\lim_{n\to\infty}\ln\frac{n}{5n+2} = \ln\frac{1}{5} \neq 0$

diverges!

$\displaystyle\sum \ln\frac{n}{n+1} = \ln(n)-\ln(n+1)$ telescoping. see class notes

Geometric series: $\sum_{n=1}^{\infty} ar^{n-1}$ will converge if $|r|<1$
will diverge if $|r|>1$

moreover, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $|r|<1$

$\boxed{\text{sum}}$

c.) $\sum_{n=1}^{\infty} 2\left(\frac{5}{7}\right)^{n-1}$ $r=\frac{5}{7}$

$|r|<1$ will converge.

$$\sum_{n=1}^{\infty} 2\left(\frac{5}{7}\right)^{n-1} = \frac{a}{1-r} = \frac{2}{1-\frac{5}{7}}\cdot\frac{7}{7} = \frac{14}{7-5} = \frac{14}{2} = 7$$

d.) $\sum_{n=1}^{\infty} 3\left(\frac{2}{5}\right)^{n}$ $r=\frac{2}{5}$, $|r|<1$ will converge

$$\sum_{n=1}^{\infty} 3\left(\frac{2}{5}\right)^{n} = \sum_{n=1}^{\infty} \underbrace{3\cdot\frac{2}{5}}_{a}\Big(\underbrace{\frac{2}{5}}_{r}\Big)^{n-1} = \frac{a}{1-r} = \frac{\frac{6}{5}}{1-\frac{2}{5}}\cdot\frac{5}{5} = \frac{6}{3} = 2$$

added example

$$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n+2} = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}_{a}\Big(\underbrace{\frac{1}{3}}_{r}\Big)^{n}$$

$$= \frac{\frac{1}{9}}{1-\frac{1}{3}}\cdot\frac{9}{9} = \frac{1}{9-3} = \frac{1}{6}$$

$2^{3n} = (2^3)^n = 8^n$

e.) $\sum_{n=0}^{\infty} \frac{2^{3n}}{(-5)^{n+1}} = \sum_{n=0}^{\infty} \frac{8^n}{(-5)(-5)^n}$

$$= \sum_{n=0}^{\infty} -\frac{1}{5}\left(-\frac{8}{5}\right)^n$$ $r=-\frac{8}{5}$ $|r|>1$ will diverge!

f.) $\sum_{n=0}^{\infty} \frac{(-1)^n+3^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} + \sum_{n=0}^{\infty} \frac{3^n}{4^n}$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n + \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = \frac{1}{1+\frac{1}{4}} + \frac{1}{1-\frac{3}{4}}$$

$r=-\frac{1}{4}$ ✓ $r=\frac{3}{4}$ ✓ $= \frac{4}{5} + 4 = \frac{24}{5}$

g.) $2-\frac{4}{7}+\frac{8}{49}-\frac{16}{343}+\ldots$ $2 - \frac{2^2}{7} + \frac{2^3}{7^2} - \frac{2^4}{7^3} + \ldots$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{7^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 2\cdot 2^n}{7^n} = \sum_{n=0}^{\infty} 2\left(-\frac{2}{7}\right)^n$$

$$= \frac{2}{1+\frac{2}{7}} = \frac{14}{9}$$

Find the values of x for which $\sum_{n=1}^{\infty} (x-3)^n$ converges. For these values of x, find the sum.

$-1 < x-3 < 1$

$\boxed{2 < x < 4}$

$$\sum_{n=1}^{\infty} (x-3)^n = \sum_{n=1}^{\infty} \underbrace{(x-3)}_{a}\underbrace{(x-3)}_{r}^{n-1}$$

sum

$$= \frac{a}{1-r} = \frac{x-3}{1-(x-3)} = \boxed{\frac{x-3}{4-x}}$$