Section 10.3

1. Determine whether the following series converge or diverge.

a.) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

TD: $\lim_{n\to\infty} \frac{1}{n\sqrt{\ln n}} = 0$ test fails.

positive series and easy to integrate, use $\boxed{\text{integral test}}$ name of test

$u = \ln x$

$du = \frac{dx}{x}$

$\int \frac{du}{\sqrt{u}} = 2\sqrt{u}$

$= 2\sqrt{\ln x}$

apply test

$$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}}$$

$$= 2\sqrt{\ln x}\Big|_2^{\infty} = 2\sqrt{\ln \infty} - 2\sqrt{\ln 2} = \infty$$

series diverges

conclusion

b.) $\sum_{n=2}^{\infty} n^2 e^{-n^3}$

TD $\lim_{n\to\infty} n^2 e^{-n^3} = \lim_{n\to\infty} \frac{n^2}{e^{n^3}} = 0$

fails.

positive series & easy to integrate.

$$\int_2^{\infty} x^2 e^{-x^3}\, dx$$

$u = -x^3$

$du = -3x^2\,dx$

$-\frac{1}{3}\int e^u\, du$

$-\frac{1}{3}e^u = -\frac{1}{3}e^{-x^3}$

$$= -\frac{1}{3}e^{-x^3}\Big|_2^{\infty} = -\frac{1}{3e^{x^3}}\Big|_2^{\infty} = \frac{-1}{3e^{\infty}} + \frac{1}{3e^8} = \frac{1}{3e^8}$$

series converges

c.) $\sum_{n=1}^{\infty} \frac{n^4}{n^{10}+n+1}$ TD fails

positive terms but not easy to integrate.

① CT $\sum_{n=1}^{\infty} \frac{n^4}{n^{10}+n+1} \leq \sum_{n=1}^{\infty} \frac{n^4}{n^{10}} = \sum_{n=1}^{\infty} \frac{1}{n^6}$

convergent p-series $p=6$

larger converges so does smaller.

common mistake: ~~$\frac{n^4}{n^{10}+n+1} \leq \frac{1}{n^6}$~~ ← convergent p-series

missing summation!

d.) $\sum_{n=17}^{\infty} \frac{1}{\sqrt{n}-4}$ TD fails

positive terms not easy to integrate

CT $\sum_{n=17}^{\infty} \frac{1}{\sqrt{n}-4} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

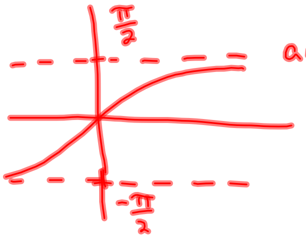
divergent p-series $p=\frac{1}{2}$

smaller series diverges so does larger.

e.) $\sum_{n=1}^{\infty} \frac{\arctan n}{n\sqrt{n}}$

TD fails

positive terms not easy to integrate



arctanx

$$\sum_{n=1}^{\infty} \frac{\arctan n}{n\sqrt{n}} \le \sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n\sqrt{n}}$$

convergent p-series $p = \frac{3}{2}$

larger conv so does smaller.

f.) $\sum_{n=1}^{\infty} \frac{n^2 - n}{n^3 + 7n}$

TD fails

positive terms

no integral

① $\sum_{n=1}^{\infty} \frac{n^2 - n}{n^3 + 7n} \le \sum_{n=1}^{\infty} \frac{n^2}{n^3}$

$= \sum_{n=1}^{\infty} \frac{1}{n}$

larger diverges test fails.

LCT If $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ and finite then both series do the same thing.

$$\lim_{n\to\infty} \frac{\frac{n^2 - n}{n^3 + 7n}\left(\frac{n}{1}\right)}{\cancel{\frac{1}{n}}} = \lim_{n\to\infty} \frac{n^3 - n^2}{n^3 + 7n} = 1$$

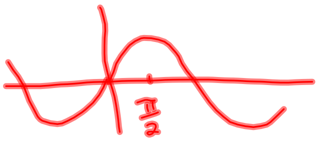
both series diverge since $\sum \frac{1}{n}$ diverges.

g.) $\sum_{n=2}^{\infty} \frac{1}{\ln n} \geq \sum_{n=2}^{\infty} \frac{1}{n}$

divergent p-series $p=1$

smaller div so does larger

h.) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$



$\sin x > 0$ for $0 < x < 1$

TD $\lim_{n\to\infty} \sin\left(\frac{1}{n^2}\right) = \sin 0$

$= 0$

Fails

$\sin\left(\frac{1}{n^2}\right)$ is positive

LCT $\sum b_n = \sum \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n^2}}{\frac{1}{n^2}}$$

recall: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$= 1 > 0$ both series converge since $\sum \frac{1}{n^2}$ converged.

2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

a.) Find the sum of the first 5 terms.

b.) Estimate the error in using the sum of the first 5 terms to approximate the sum of the series.

c.) Find the sum correct to 10 decimal places.

a.) $S_5 = 1 + \frac{1}{2^3} + \cdots + \frac{1}{5^3}$

b.) positive series, use $R_n \leq \int_n^{\infty} f(x)\,dx$

here, $n=5$

$$R_5 \leq \int_5^{\infty} \frac{dx}{x^3}$$

$$R_5 \leq -\frac{1}{2x^2}\Big|_5^{\infty}$$

$$R_5 \leq \cancel{-\frac{1}{2(\infty)^2}}^{\,0} + \frac{1}{2(5)^2}$$

$$\boxed{R_5 \leq \frac{1}{50}}$$

c.) ① find n so that $R_n \leq \frac{1}{10^{10}}$

② $S_n = ?$

solve $\underbrace{R_n \leq \int_n^{\infty} \frac{dx}{x^3} \leq \frac{1}{10^{10}}}$

$$-\frac{1}{2x^2}\Big|_n^{\infty} \leq \frac{1}{10^{10}}$$

$$\cancel{-\frac{1}{2(\infty)^2}}^{\,0} + \frac{1}{2n^2} \leq \frac{1}{10^{10}} \rightarrow \frac{1}{2n^2} \leq \frac{1}{10^{10}}$$

$$\frac{10^{10}}{2} \leq n^2$$

$$70710.6 \leq n$$

$$n \geq 70711$$

$$\boxed{S_{70711} = 1 + \frac{1}{2^3} + \cdots + \frac{1}{70711^3}}$$

3. Consider $\sum_{n=1}^{\infty} \frac{\cos^2 n + 2}{n^5}$

a.) Prove the series converges.

b.) Approximate the sum of the series using s_6.

c.) By comparing the series to a p-series, estimate the error in using s_6 to approximate the sum of the series.

a.) $\sum_{n=1}^{\infty} \frac{\cos^2 n + 2}{n^5} \leq \sum_{n=1}^{\infty} \frac{3}{n^5}$

convergent p-series $p=5$
larger conv so does smaller.

b.) $S_6 = \frac{\cos^2 1 + 2}{1^5} + \dots + \frac{\cos^2 6 + 2}{6^5}$

$R_6 \leq \int_6^{\infty} \frac{3}{x^5}\,dx$

$= -\frac{3}{4x^4}\Big|_6^{\infty} = \boxed{0 + \frac{3}{4(6^4)}}$

4. How many terms of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ would you need to add to find its sum to within 0.01?

Since $\frac{1}{n(\ln n)^2} > 0$ use $R_n \leq \underbrace{\int_n^\infty f(x)\,dx \leq \frac{1}{100}}$

$$\int_n^\infty \frac{dx}{x(\ln x)^2} \leq \frac{1}{100}$$

$u = \ln x$

$du = \frac{dx}{x}$

$$\int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\ln x}$$

$$\left.\frac{-1}{\ln x}\right|_n^\infty \leq \frac{1}{100}$$

$$0 + \frac{1}{\ln n} \leq \frac{1}{100}$$

$$100 \leq \ln n$$

$$e^{100} \leq n$$

use at least

$e^{100} - 1$

since series starts at 2

Section 10.4

5. Use the alternating series test to determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges.

AST

TD $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n+1}} = 0$ fails.

AST: show $\left\{ \frac{1}{\sqrt{n+1}} \right\}$ satisfies ① $a_{n+1} \le a_n$ ② $\lim_{n\to\infty} a_n = 0$

① $\frac{1}{\sqrt{n+2}} \le \frac{1}{\sqrt{n+1}} \to a_{n+1} \le a_n$ ✓

② $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ ✓

$\boxed{\text{series converges by AST}}$

6. Determine whether the following series converge absolutely, converge conditionally, or diverge.

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\sqrt{n}}$

Test absolut convergence first: $\to$ does $\sum |a_n|$ converge?

Look at $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}}$

convergent p-series $p = \frac{5}{2}$

absolutely convergent.

b.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ TD fails

absolute convergence

Test for AC look at $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergent series.

"throw off $(-1)^n$"

so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is not AC

check for conditional convergence by using AST

show $\left\{ \frac{1}{\sqrt{n}} \right\}$ ① $a_{n+1} \le a_n$ ② $\lim_{n\to\infty} a_n = 0$

① $\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}}$ ✓

② $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

series converges but not absolutely (conditional conv)

c.) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ TD fails

Test for AC $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

u-sub $u = \ln x$

Look at $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln x}\Big|_2^{\infty}$

$= -\cancel{\frac{1}{\infty}}^{\,0} + \frac{1}{\ln 2}$

$= \frac{1}{\ln 2}$

$\boxed{\text{series converges absolutely done!}}$

d.) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

T.D. $\lim_{n\to\infty} \frac{(-1)^n n}{n+1} \neq 0$

$\boxed{\text{series diverges}}$

e.) $\sum_{n=1}^{\infty} \frac{n^2}{(-4)^n}$ TD $\lim_{n\to\infty} \frac{n^2}{(-4)^n} = 0$ Fails

series contains an exponential $(-4)^n$ use RT

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \begin{cases} <1 & \text{AC} \\ >1 & \text{D} \\ =1 & \text{fails} \end{cases}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(-4)^{n+1}}\cdot\frac{(-4)^n}{n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(-4)\cancel{(-4)^n}}\cdot\frac{\cancel{(-4)^n}}{n^2}\right|$$

$$= \lim_{n\to\infty}\left|-\frac{1}{4}\,\frac{\cancel{(n+1)^2}}{\cancel{n^2}}\right|$$

$$= \left|-\frac{1}{4}\right| = \frac{1}{4} < 1 \quad \boxed{AC}$$

f.) $\sum_{n=1}^{\infty} \frac{3^n n^2}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(n+1)^2}{(2n+2)!}\cdot\frac{(2n)!}{3^n n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3\cdot\cancel{3^n}(n+1)^2}{(2n+2)(2n+1)\cancel{(2n)!}}\cdot\frac{\cancel{(2n)!}}{\cancel{3^n}n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3(n+1)^2}{(2n+2)(2n+1)n^2}\right| = 0 < 1$$

converges absolutely

7. Show $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$ converges absolutely and then approximate the sum of the series with the third partial sum, s_2. How close is this approximation to the sum of the series?

① prove AC by RT

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)\cancel{(-1)^n}}{(2n+3)(2n+2)\cancel{(2n+1)!}} \frac{\cancel{(2n+1)!}}{\cancel{(-1)^n}} \right|$$

$$= 0 < 1 \rightarrow \boxed{AC}$$

three terms, hence third partial sum

② $S_2 = a_0 + a_1 + a_2$

$$\boxed{S_2 = 1 - \frac{1}{3!} + \frac{1}{5!}}$$

③ alternating series → use $|R_n| \leq |a_{n+1}|$ $n=2$

$$|R_2| \leq |a_3| = \left| \frac{(-1)^3}{7!} \right| = \boxed{\frac{1}{7!}}$$

8. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ correct to within 3 decimal places.

① Find n so that $|R_n| \le \underbrace{|a_{n+1}| \le 10^{-3}}$

$$\frac{1}{(n+1)^2} \le \frac{1}{10^3}$$

$$10^3 \le (n+1)^2$$

$$\sqrt{10^3} \le n+1$$

$$\sqrt{10^3} - 1 \le n$$

$$31.62 \le n$$

n must be at least 32.

② $\boxed{S_{32} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \cdots \frac{1}{32^2}}$