Section 10.5

1. For the following power series, find the radius and interval of convergence.

a.) $\sum_{n=0}^{\infty} n^2(x+5)^n$

RT $\lim_{n\to\infty} \left| \frac{(n+1)^2 (x+5)^{n+1}}{n^2 (x+5)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)^2 (x+5) \cancel{(x+5)^n}}{n^2 \cancel{(x+5)^n}} \right|$

$= \lim_{n\to\infty} \left| (x+5) \cancel{\frac{(n+1)^2}{n^2}} \right|$ $\to 1$

$= |x+5| < 1$

$\boxed{R=1}$

$-6 \quad -5 \quad -4$

test endpoints: $\sum_{n=0}^{\infty} n^2 (x+5)^n$

$x=-4$: $\sum_{n=0}^{\infty} n^2 (1)^n$

diverges by TD
$\lim_{n\to\infty} n^2 \neq 0$

$x=-6$: $\sum_{n=0}^{\infty} n^2 (-1)^n$

diverges by TD
$\lim_{n\to\infty} n^2 (-1)^n \neq 0$

$I = (-6, -4)$

or $-6 < x < -4$

b.) $\sum_{n=0}^{\infty} \frac{2^n(x+1)^n}{n^2+2}$

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(x+1)^{n+1}}{(n+1)^2+2} \cdot \frac{n^2+2}{2^n(x+1)^n} \right|$$

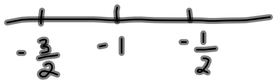
$$\lim_{n\to\infty} \left| \frac{2\cdot\cancel{2^n}(x+1)\cancel{(x+1)^n}}{(n+1)^2+2} \cdot \frac{n^2+2}{\cancel{2^n}\cancel{(x+1)^n}} \right|$$

$$\lim_{n\to\infty} \left| 2(x+1)\overbrace{\frac{(n^2+2)}{(n+1)^2+2}}^{\to 1} \right| = |2(x+1)| < 1$$

$$|x+1| < \frac{1}{2} \qquad \boxed{R = \frac{1}{2}}$$

b.) $\sum_{n=0}^{\infty} \frac{2^n(x+1)^n}{n^2+2}$

Test endpoints



$$x=-\frac{1}{2}: \quad \sum_{n=0}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{n^2+2} = \boxed{\sum_{n=0}^{\infty}} \frac{1}{n^2+2} \leq \underbrace{\boxed{\sum} \frac{1}{n^2}}$$

converges by p-series $p=2$

larger series conv, so does smaller. ✓

$$x=-\frac{3}{2}: \quad \sum_{n=0}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{n^2+2} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+2}}_{AST}$$

$$\frac{1}{(n+1)^2+2} < \frac{1}{n^2+2} \checkmark$$

$$\lim_{n\to\infty} \frac{1}{n^2+2} = 0 \checkmark$$

converges by AST

$$\boxed{I = \left[-\frac{3}{2}, -\frac{1}{2}\right]}$$

c.) $\sum_{n=2}^{\infty} \frac{(2x-1)^n}{4^n \ln n}$

$$\lim_{n\to\infty} \left| \frac{(2x-1)^{n+1}}{4^{n+1} \ln(n+1)} \cdot \frac{4^n \ln n}{(2x-1)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{2x-1}{4 \ln(n+1)} \cdot \ln n \right| = \lim_{n\to\infty} \left| \frac{2x-1}{4} \frac{\ln n}{\ln(n+1)} \right|$$

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \quad \frac{\infty}{\infty}$$

$$\overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}}$$

$$= \lim_{n\to\infty} \frac{n+1}{n} = 1$$

$$\left| \frac{2x-1}{4} \right| < 1$$

$$|2x-1| < 4 \longrightarrow \left| x - \frac{1}{2} \right| < 2$$

$-\frac{3}{2}$ $\quad \frac{1}{2}$ $\quad \frac{5}{2}$

c.) $\sum_{n=2}^{\infty} \frac{(2x-1)^n}{4^n \ln n}$ test endpoints: $x = \frac{5}{2}$ $\sum_{n=2}^{\infty} \frac{\cancel{4^n}}{\cancel{4^n} \ln n}$

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} > \underbrace{\sum_{n=2}^{\infty} \frac{1}{n}}$$

divergent p-series $p=1$

smaller div. so does larger.

$$x = -\frac{3}{2}: \sum_{n=2}^{\infty} \frac{(-4)^n}{4^n \ln n}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

AST

$$\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$$

$$\lim_{n\to\infty} \frac{1}{\ln n} = 0$$

converges by AST

$$\boxed{I = \left[-\frac{3}{2}, \frac{5}{2}\right)}$$

d.) $\sum_{n=0}^{\infty} \frac{n!(x+2)^{n-1}}{5^{n-1}}$

$(n+1)\cancel{n!}$ $\quad (x+2)\cancel{(x+2)^{n-1}}$

RT $\quad \lim_{n\to\infty} \left| \frac{(n+1)!\,(x+2)^n}{5^n} \cdot \frac{\cancel{5^{n-1}}}{\cancel{n!}\,\cancel{(x+2)^{n-1}}} \right|$ $\quad 5 \cdot \cancel{5^{n-1}}$

$$\lim_{n\to\infty} \left| \frac{(n+1)(x+2)}{5} \right| = \infty \quad \text{unless} \quad \underline{\underline{x=-2}}$$

$$\boxed{I = \{-2\}, \quad R = 0}$$

e.) $\sum_{n=0}^{\infty} \frac{(-3)^n (x+3)^n}{n!}$

$$\underline{\underline{RT}} \quad \lim_{n\to\infty} \left| \frac{(-3)^{n+1}(x+3)^{n+1}}{(n+1)!} \cdot \frac{\cancel{n!}}{(-3)^n (x+3)^n} \right|$$

$$(n+1)\cancel{n!}$$

$$\lim_{n\to\infty} \left| \frac{(-3)(x+3)}{n+1} \right| = 0 \quad \text{for } \underline{\underline{\text{all}}} \ x$$

$$\boxed{I = (-\infty, \infty) \qquad R = \infty}$$

2. Suppose it is known that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series:

$\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$

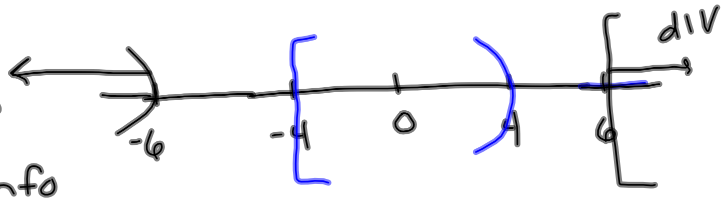
diverges when $x = 6$.

a.) $\sum_{n=0}^{\infty} c_n (2)^n$ conv

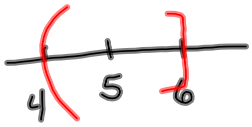
b.) $\sum_{n=0}^{\infty} c_n (8)^n$ div

c.) $\sum_{n=0}^{\infty} c_n (4)^n$ not enough info

d.) $\sum_{n=0}^{\infty} c_n (-5)^n$ not enough info



3. Suppose it is known that $\sum_{n=0}^{\infty} c_n(x-5)^n$ converges when $x = 6$. On what interval are we guaranteed convergence?



$(4, 6]$

4. Express the following functions as a $\boxed{\text{power series.}}$ $\longrightarrow$ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$

Identify the radius and interval of convergence.

a.) $f(x) = \dfrac{1}{2+5x^2}$

$$= \frac{1}{2\left(1+\frac{5x^2}{2}\right)} = \frac{1}{2\left(1-\left(-\frac{5x^2}{2}\right)\right)} = \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{5x^2}{2}\right)^n$$

where $\left|-\frac{5x^2}{2}\right| < 1$

$|x^2| < \frac{2}{5}$

$|x| < \sqrt{\frac{2}{5}}$

R

$$= \sum_{n=0}^{\infty} \frac{1}{2} \frac{(-1)^n 5^n x^{2n}}{2^n}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^{2n}}{2^{n+1}}}$$

b.) $f(x) = \ln(x+4)$

$$\frac{d}{dx} \ln(x+4) = \frac{1}{x+4} = \frac{1}{4\left(1+\frac{x}{4}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4}\left(-\frac{x}{4}\right)^n, \quad \left|-\frac{x}{4}\right| < 1 \quad \boxed{R=4}$$

$$\int \frac{1}{x+4} = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^{n+1}}$$

$$\ln(x+4) = \xcancel{C}^{\ln 4} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{4^{n+1}(n+1)}$$

let $x=0$ to find C: $\ln 4 = C$

c.) $f(x) = x\ln(x+4)$

$$x\ln(x+4) = x\left(\ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{4^{n+1}(n+1)}\right)$$

d.) $f(x) = \arctan(2x^3)$

$$\frac{d}{dx} \arctan(2x^3) = \frac{6x^2}{1+4x^6}$$

$$= 6x^2 \sum_{n=0}^{\infty} (-4x^6)^n, \quad |-4x^6| < 1$$

$$|x| < \sqrt[6]{\frac{1}{4}}$$

$$= 6x^2 \sum_{n=0}^{\infty} (-4)^n x^{6n}$$

$$\int \frac{d}{dx} \arctan(2x^3) = \int 6 \sum_{n=0}^{\infty} (-4)^n x^{6n+2}$$

$$\arctan(2x^3) = \cancel{C}^{0} + 6 \sum_{n=0}^{\infty} \frac{(-4)^n x^{6n+3}}{6n+3}$$

$x=0$: $\arctan(0) = C \rightarrow C = 0$

e.) $f(x) = \frac{x}{(1-x^2)^2}$

$$f(x) = \frac{x}{(1-2x)^2} = x\left(\frac{1}{(1-2x)^2}\right)$$

$$\int \frac{1}{(1-2x)^2}\,dx = \frac{1}{2} \cdot \frac{1}{1-2x} = \frac{1}{2} \sum_{n=0}^{\infty} (2x)^n \qquad \boxed{R = \frac{1}{2}}$$

$$\frac{d}{dx} \int \frac{dx}{(1-2x)^2} = \frac{d}{dx} \sum_{n=0}^{\infty} 2^{n-1} x^n$$

$$\frac{1}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1}$$

$$\frac{x}{(1-2x)^2} = x \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1}$$

$$= \sum_{n=1}^{\infty} 2^{n-1} n x^n$$

5. Express $\int_0^{0.1} \frac{1}{1+x^5}\,dx$ as an infinite series. Use the sum of the first 3 terms of this series to approximate $\int_0^{0.1} \frac{1}{1+x^5}\,dx$.

$$\int_0^{0.1} \frac{1}{1+x^5}\,dx = \int_0^{0.1} \sum_{n=0}^{\infty} (-x^5)^n\,dx$$

$$= \int_0^{0.1} \sum_{n=0}^{\infty} (-1)^n x^{5n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{5n+1} \Bigg|_0^{0.1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (0.1)^{5n+1}}{5n+1} - \cancel{\Sigma}\, 0 \to 0$$

sum of first three terms is $a_0 + a_1 + a_2$

$$\frac{(-1)^0 (0.1)^1}{1} + \frac{-1(0.1)^6}{6} + \frac{(0.1)^{11}}{11}$$

Bound on remainder $|R_n| \le |a_{n+1}|$

$$|R_2| \le |a_3| = \frac{(.1)^{16}}{16}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^n n!}{(n+1)n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$$ converges by AST

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{7n^4}$$

how many terms do we need to add so that $|\text{error}| < 0.00005$?

use $|R_n| < \underbrace{|a_{n+1}| < 0.00005}$

$$\frac{1}{7(n+1)^4} < 0.00005$$

$$\frac{d}{dx} \ln(x^2+2) = \frac{2x}{x^2+2} = \frac{\cancel{2}x}{\cancel{2}(1+\frac{x^2}{2})}$$

$$= x \sum_{n=0}^{\infty} \left(-\frac{x^2}{2}\right)^n$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n}$$

$$\frac{d}{dx} \ln(x^2+2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n}$$

$$\ln(x^2+2) = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2^n(2n+3)}$$

$x=0$: $\ln 2 = C$