Problem 1 (Roots, derivatives, and integrals of implicit functions).
Define $f(x)$ implicitly as follows: for $x \geq 1$, let $f(x) = \sin y$ where $y$ is the solution of

$$y \log y = x$$

for every $x$ given as argument to $f(\cdot)$. (See homework 9 for a similar example, but note that here $f(x) = \sin y$, not just $f(x) = y$.)

Implement a computer routine that can compute $f(x)$ when given $x$. With this, do the following things:

a) Plot $f(x)$ in the range $x = 1 \ldots 40$;

b) Find the locations $x_1, \ldots, x_4$ of the first four zeros (roots) of $f(x)$, using your method of choice, to an accuracy of at least four digits;

c) Compute $f'(10)$ using your method of choice, to an accuracy of at least four digits;

d) Compute $\int_1^{40} f(x) \, dx$ using your method of choice, to an accuracy of at least four digits. \hspace{5cm} (8 \text{ points})
Problem 2 (High accuracy ODE solution). Use the Runge-Kutta method of order 4 to solve the following trivial system:
\[ x'(t) = x(t), \quad x(0) = 1. \]
The solution of this is, of course, \( x(t) = e^t \). Choose a sequence of step sizes \( h \) to compute approximate \( x(1) = e \) numerically. Demonstrate, by choosing \( h \) smaller and smaller (for example \( h = \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{128} \)), the convergence order of the RK4 method. If you continue to choose \( h \) smaller and smaller (beyond reasonable limits), how small can you get the error? Explain why you can’t get it smaller than that.

(3 points)

Problem 3 (Planetary motion). Small objects (such as planets), revolve around massive objects (such as the sun) based on the interplay between gravity and their desire to move on straight lines. The equations of motion are as follows:
\[
\begin{align*}
 x''(t) &= -\frac{GMx(t)}{(x(t)^2 + y(t)^2)^{3/2}}, \\
 y''(t) &= -\frac{GMy(t)}{(x(t)^2 + y(t)^2)^{3/2}}, \\
 x(0) &= x_0, \\
 y(0) &= y_0, \\
 x'(0) &= u_0, \\
 y'(0) &= v_0.
\end{align*}
\]
Here, \( G = 6.67 \cdot 10^{-11} \) is the gravitational constant, and \( M = 1.9891 \cdot 10^{30} \) is the mass of the sun.

Answer the following questions:

a) Convert the equations to ODE standard form;

b) Explain (in words, without implementing anything), which of the methods for ODE solvers we have discussed in class (i.e., Taylor expansion of order 1 and 2, explicit and implicit Euler, Crank-Nicolson, multistep methods, and Runge-Kutta methods) is probably going to be best suited to this problem and why. Compare its advantages against the other methods;

c) Implement one of the methods (not necessarily the one you think is best suited, if you don’t feel up to the task of actually implementing it for this problem). Use the following initial conditions:
\[
\begin{align*}
 x_0 &= 1.5 \cdot 10^{11}, & u_0 &= 0, \\
 y_0 &= 0, & v_0 &= 3000.
\end{align*}
\]
(This is the correct location of the earth, but its velocity is 10 times smaller than it really is.) Compute and plot \( x(t), y(t) \) up to time \( T = 3 \cdot 10^7 \). Determine the closest distance to the sun, i.e. the minimal value of \( \sqrt{x(t)^2 + y(t)^2} \) within the time interval \( 0 \leq t \leq T \) as accurate as you can (three digits is enough; in any case, you have to justify why you think your digits are correct).

(7 points)
Problem 4 (Two-dimensional root finding/Newton’s method). Use Newton’s method to find the minimum closest to (2, 3) of the function

\[ g(x, y) = \frac{e^x e^y}{x^2 y^3}. \]

Start your iteration at \((x_0, y_0) = (1, 1)\). Compute an accurate approximation \((\hat{x}, \hat{y})\) to the location of the minimum. Generate a table or graph in which you show the distance of iterate \(k\) to your best guess of the true location, i.e. show \(\sqrt{(x_k - \hat{x})^2 + (y_k - \hat{y})^2}\) as a function of \(k\). Give an interpretation of this table or graph. (3 points)