MATH 609-602: Numerical Methods

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Test 1 – due Tuesday 10/4/2005

I hereby certify that I have prepared my answers alone and without help by others:

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(signature of student)

The usual rules of academic intregrity apply. In particular, the Aggie Honor Code "An Aggie does not lie, cheat or steal, or tolerate those who do" should be selfevident, see

`http://www.tamu.edu/aggiehonor.html`

Problem 1 (Taylor series expansion of x^{-1}.) Derive the (infinite, without remainder term) Taylor series of

$$f(x) = x^{-1},$$

when expanded around $x_0 = 1$. Determine for which values in the range $0 \leq x \leq \infty$ this series actually converges, i.e. for which the Taylor series up to term k converges against $f(x)$ as $k \to \infty$. Does it converge for negative values $x < 0$?

(3 points)

Problem 2 (Taylor series expansion of A^{-1}). For real x with $|1-x| < 1$, the following formula is true:

$$\frac{1}{x} = \sum_{k=0}^{\infty} (1-x)^k.$$

One could suspect that the same holds for invertible matrices $X \in \mathbb{R}^{n\times n}$, so that

$$X^{-1} = \sum_{k=0}^{\infty} (I - X)^k,$$

at least if $\|I - X\| < 1$ for some matrix norm.

If D is the diagonal of a matrix A, then we have shown in connection with Jacobi iteration that $\|I - D^{-1}A\|_\infty < 1$ if A is a strongly diagonally dominant matrix. Let us therefore consider $X = D^{-1}A$. We then have

$$(D^{-1}A)^{-1} = \sum_{k=0}^{\infty} (I - D^{-1}A)^k,$$

or

$$A^{-1} = \sum_{k=0}^{\infty} (I - D^{-1}A)^k D^{-1},$$

Let A, b be the 100×100 matrix and 100-dimensional vector defined by

$$A_{ij} = \begin{cases} 2.01 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise,} \end{cases} \qquad b_i = \frac{1}{100} \sin\left(\frac{2\pi i}{50}\right).$$

Obviously this matrix is strongly diagonally dominant. Compute approximations to $x = A^{-1}b$ by

$$x_N = \sum_{k=0}^{N} (I - D^{-1}A)^k D^{-1} b, \tag{1}$$

for $N = 0, 2, 5, 10, 20, 50, 100, 200$. Do they converge? Plot $\|x_N - x_{200}\|$ for these vectors, and graph $(x_N)_i$ against i as in Problem 3 of Homework 4.

Using equation (1), find a recursion formula that expresses x_N in terms of x_{N-1} for $N > 0$, and state the initial vector x_0. Compare the recursion formula with that of the Jacobi iteration.

(6 points)

Problem 3 (Matrix norms). Just as for vector norms, all possible norms for matrices are equivalent up to a constant.

a) Consider the 2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for which all entries are positive, i.e. $a, b, c, d \geq 0$. State under which conditions $\|A\|_1 = 1$. Among the matrices with all positive entries and with $\|A\|_1 = 1$, which matrix has the largest infinity norm, $\|A\|_\infty$?

b) If you know that the inequalities

$$c\|v\|_q \leq \|v\|_p \leq C\|v\|_q$$

hold for two vector norms $\|\cdot\|_p, \|\cdot\|_q$ with $1 \leq p, q \leq \infty$, can you infer that a similar relationship

$$c'\|A\|_q \leq \|A\|_p \leq C'\|A\|_q$$

with different constants c', C' holds for the corresponding matrix norm

$$\|A\|_p = \max_v \frac{\|Av\|_p}{\|v\|_p}$$

(and the corresponding definition of the q-matrix norm)?

(4 points)

Problem 4 (Newton's method in 2d). Consider the function

$$g(x, y) = (x^2 - 1)^2 + y^2.$$

Its two minima obviously lie at $x = \pm 1, y = 0$.

a) Use a program for a two-dimensional Newton method to find the minima of $g(x, y)$ and demonstrate convergence of iterates X_k towards one of these minima when started from $X_0 = (x_0, y_0) = (2, 1)$ and when started from $X_0 = (x_0, y_0) = (-2, 1)$.

b) Using only paper and pencil, show what happens if one starts from $X_0 = (x_0, y_0) = (0, 2)$. Is the point to which the algorithm converges a minimum of $g(x, y)$? If not, what is happening? Is the Newton method for finding minima broken? (Hint: You should feel free to help your imagination by letting the program from part a) run for this choice of initial values X_0 and seeing where it converges. It is also always helpful to visualize functions for which you are looking for a minimum.)

(5 points)