Problem 1 (Taylor series expansion of $x^{-1}$.) Derive the (infinite, without remainder term) Taylor series of

$$f(x) = x^{-1},$$

when expanded around $x_0 = 1$. Determine for which values in the range $0 \leq x \leq \infty$ this series actually converges, i.e. for which the Taylor series up to term $k$ converges against $f(x)$ as $k \to \infty$. Does it converge for negative values $x < 0$?

(3 points)

Problem 2 (Taylor series expansion of $A^{-1}$). For real $x$ with $|1-x| < 1$, the following formula is true:

$$\frac{1}{x} = \sum_{k=0}^{\infty} (1-x)^k.$$
One could suspect that the same holds for invertible matrices $X \in \mathbb{R}^{n \times n}$, so that

$$X^{-1} = \sum_{k=0}^{\infty} (I - X)^k,$$

at least if $\|I - X\| < 1$ for some matrix norm.

If $D$ is the diagonal of a matrix $A$, then we have shown in connection with Jacobi iteration that $\|I - D^{-1}A\|_\infty < 1$ if $A$ is a strongly diagonally dominant matrix. Let us therefore consider $X = D^{-1}A$. We then have

$$(D^{-1}A)^{-1} = \sum_{k=0}^{\infty} (I - D^{-1}A)^k,$$

or

$$A^{-1} = \sum_{k=0}^{\infty} (I - D^{-1}A)^k D^{-1},$$

Let $A, b$ be the $100 \times 100$ matrix and 100-dimensional vector defined by

$$A_{ij} = \begin{cases} 2.01 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise}, \end{cases} \quad b_i = \frac{1}{100} \sin \left( \frac{2\pi i}{50} \right).$$

Obviously this matrix is strongly diagonally dominant. Compute approximations to $x = A^{-1}b$ by

$$x_N = \sum_{k=0}^{N} (I - D^{-1}A)^k D^{-1}b,$$

for $N = 0, 2, 5, 10, 20, 50, 100, 200$. Do they converge? Plot $\|x_N - x_{200}\|$ for these vectors, and graph $(x_N)_i$ against $i$ as in Problem 3 of Homework 4.

Using equation (1), find a recursion formula that expresses $x_N$ in terms of $x_{N-1}$ for $N > 0$, and state the initial vector $x_0$. Compare the recursion formula with that of the Jacobi iteration.

(6 points)

Problem 3 (Matrix norms). Just as for vector norms, all possible norms for matrices are equivalent up to a constant.

a) Consider the $2 \times 2$ matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for which all entries are positive, i.e. $a, b, c, d \geq 0$. State under which conditions $\|A\|_1 = 1$. Among the matrices with all positive entries and with $\|A\|_1 = 1$, which matrix has the largest infinity norm, $\|A\|_\infty$?
b) If you know that the inequalities
\[ c\|v\|_q \leq \|v\|_p \leq C\|v\|_q \]
hold for two vector norms \( \| \cdot \|_p, \| \cdot \|_q \) with \( 1 \leq p, q \leq \infty \), can you infer that a similar relationship
\[ c'\|A\|_q \leq \|A\|_p \leq C'\|A\|_q \]
with different constants \( c', C' \) holds for the corresponding matrix norm
\[ \|A\|_p = \max_v \frac{\|Av\|_p}{\|v\|_p} \]
(and the corresponding definition of the \( q \)-matrix norm)?

(4 points)

**Problem 4 (Newton’s method in 2d).** Consider the function
\[ g(x, y) = (x^2 - 1)^2 + y^2. \]
Its two minima obviously lie at \( x = \pm 1, y = 0 \).

a) Use a program for a two-dimensional Newton method to find the minima of \( g(x, y) \) and demonstrate convergence of iterates \( X_k \) towards one of these minima when started from \( X_0 = (x_0, y_0) = (2, 1) \) and when started from \( X_0 = (x_0, y_0) = (-2, 1) \).

b) Using only paper and pencil, show what happens if one starts from \( X_0 = (x_0, y_0) = (0, 2) \). Is the point to which the algorithm converges a minimum of \( g(x, y) \)? If not, what is happening? Is the Newton method for finding minima broken? (Hint: You should feel free to help your imagination by letting the program from part a) run for this choice of initial values \( X_0 \) and seeing where it converges. It is also always helpful to visualize functions for which you are looking for a minimum.)

(5 points)