

# MATH 412: Theory of Partial Differential Equations

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## Midterm exam 10/25/2006

*Below are five problems that you should work on. This may seem a lot for 75 minutes. However, Problems 1, 2, 5a, and 5b are pretty simple. If you can see that you are running out of time, do the simpler questions first. Please don't forget to sign the disclaimer on the backside. Good luck! — WB*

**Problem 1 (Linearity).** An operator  $L$  is linear if (i)  $L(\alpha u) = \alpha L(u)$ , (ii)  $L(u + v) = L(u) + L(v)$  for all  $u, v$  and all  $\alpha \in \mathbb{R}$ . Which of the following are linear operations on a function  $f(x)$  and why/why not:

$$a) \quad L(f) = \frac{\partial^2}{\partial x^2} f(x), \quad b) \quad L(f) = \int_0^1 3 \sin(f(x)) \, dx, \quad c) \quad L(f) = \left[ \frac{\partial^4}{\partial x^4} f(x) \right]^2.$$

What is the importance of linear operators with regard to the construction of solutions if they appear in homogeneous equations? **(4 points)**

*Answer:* This problem only requires that you verify the two properties for each of the three operators. Both have to be true for the operator to be linear. Only the first operator is linear. For the second and third one, the first condition is violated and so we won't have to check the second condition any more (it is also violated). Let's show this for the third operator:

$$\begin{aligned} L(\alpha u(x)) &= \left[ \frac{\partial^4}{\partial x^4} \alpha u(x) \right]^2 = \left[ \alpha \frac{\partial^4}{\partial x^4} u(x) \right]^2 \\ &= \alpha^2 \left[ \frac{\partial^4}{\partial x^4} u(x) \right]^2 = \alpha^2 L(u(x)) \neq \alpha L(u(x)). \end{aligned}$$

So the first condition isn't satisfied and the operator can't be linear.

**Problem 2 (Solutions to the Poisson equation in 1-d).** What is the solution of the following Poisson equation in one space dimension:

$$\begin{aligned} -\frac{d^2}{dx^2} u(x) &= x^2 + \cos x && \text{for } 0 < x < 1, \\ u(0) &= 1 && \text{at the left boundary at } x = 0, \\ u(1) &= 2 && \text{at the right boundary at } x = 1. \end{aligned}$$

(2 points)

*Answer:* We can solve this problem by forming antiderivatives twice as follows:

$$\begin{aligned}\frac{d^2}{dx^2}u(x) &= -x^2 - \cos x \\ \frac{d}{dx}u(x) &= -\frac{1}{3}x^3 - \sin x + C \\ u(x) &= -\frac{1}{12}x^4 + \cos x + Cx + D.\end{aligned}$$

We then have to determine constants  $C, D$  using boundary conditions:

$$\begin{aligned}1 = u(0) &= -\frac{1}{12}(0)^4 + \cos(0) + C(0) + D = 1 + D, \\ 2 = u(1) &= -\frac{1}{12}(1)^4 + \cos(1) + C(1) + D = -\frac{1}{12} + \cos 1 + C + D.\end{aligned}$$

This is solved by  $D = 0, C = \frac{25}{12} - \cos(1)$ , i.e. the overall solution is  $u(x) = -\frac{1}{12}x^4 + \cos x + (\frac{25}{12} - \cos(1))x$ .

**Problem 3 (Energy method).** Show that solutions of the Poisson equation

$$\begin{aligned}-\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega\end{aligned}$$

are unique, using the “energy method”.

(3 points)

*Answer:* Proving uniqueness always starts like this: let us assume that there are two functions  $u_1(\mathbf{x}), u_2(\mathbf{x})$  both of which satisfy all the equations, i.e.

$$\begin{aligned}-\Delta u_1 &= f && \text{in } \Omega, \\ u_1 &= g && \text{on } \partial\Omega,\end{aligned}$$

and

$$\begin{aligned}-\Delta u_2 &= f && \text{in } \Omega, \\ u_2 &= g && \text{on } \partial\Omega.\end{aligned}$$

The next step is to introduce a function  $\delta(\mathbf{x})$  that is the difference between the two proposed solutions:  $\delta(\mathbf{x}) = u_1(\mathbf{x}) - u_2(\mathbf{x})$ . What we want to show is that there is no way that  $\delta$  is non-zero, because this then implies that for any two solutions  $u_1, u_2$ , the difference is zero – in other words that any two solutions are actually the same, i.e. that there is in fact only one, unique solution.

To do this, we first derive the equations that  $\delta$  has to satisfy:

$$\begin{aligned}-\Delta\delta &= -\Delta(u_1 - u_2) = f - f = 0 && \text{in } \Omega, \\ \delta &= u_1 - u_2 = g - g = 0 && \text{on } \partial\Omega.\end{aligned}$$

This means that the difference  $\delta$  has to satisfy the Laplace equation with zero boundary conditions. What follows from here differs between the energy method and the classical approach. In either case, we want to show that the solution to this equation must be zero. In the energy method, we multiply the first equation with  $\delta$  and integrate over  $\Omega$ :

$$\int_{\Omega} (-\Delta\delta)\delta = \int_{\Omega} 0\delta.$$

Integrating by parts on the left and noting that the right hand side is zero yields

$$\int_{\Omega} \nabla\delta \cdot \nabla\delta - \int_{\partial\Omega} (\mathbf{n} \cdot \nabla\delta)\delta = 0.$$

Because  $\delta = 0$  on  $\partial\Omega$ , the second term on the left is zero. Consequently,

$$\int_{\Omega} |\nabla\delta|^2 = 0.$$

So what we have here is that the integral over a strictly non-negative quantity is zero. That can only be if the integrand is in fact zero, i.e.  $\nabla\delta = 0$ , which implies  $\delta = \text{const}$ . On the other hand, we know that  $\delta = 0$  on the boundary. If we now assume that  $\delta$  is continuous (an assumption that is in fact not strictly true, but true enough for the present purpose – ask me for details if you’re interested), then we conclude that  $\delta = 0$  everywhere, not only on the boundary but also in the interior of  $\Omega$ . This is what we wanted to prove.

**Problem 4 (Maximum principle).** State in words what the maximum principle for the Laplace equation means. Describe how it can be used to show uniqueness of solutions of the Poisson equation

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega. \end{aligned}$$

**(3 points)**

*Answer:* The maximum principle states that for solutions  $w(\mathbf{x})$  of the Laplace equation (i.e. the Poisson equation with a zero right hand side in the PDE)

$$\begin{aligned} -\Delta w &= 0 && \text{in } \Omega, \\ w &= h && \text{on } \partial\Omega \end{aligned}$$

the maximum value of  $w$  will be attained somewhere at the boundary, i.e. that there is not going to be a point inside  $\Omega$  where  $w(\mathbf{x})$  is larger than the maximum of the boundary values  $h(\mathbf{x})$ . In other words,  $w(\mathbf{x}) \leq \max_{\mathbf{y} \in \partial\Omega} h(\mathbf{y})$  for all  $\mathbf{x} \in \Omega$ . The corresponding minimum principle states that also the opposite is true, i.e. that the solution is never smaller than the minimum value at the boundary:  $w(\mathbf{x}) \geq \min_{\mathbf{y} \in \partial\Omega} h(\mathbf{y})$  for all  $\mathbf{x} \in \Omega$ .

Now, in the previous question, we have already shown that the difference  $\delta$  between an arbitrarily chosen pair of solutions  $u_1, u_2$  satisfies the equation

$$\begin{aligned} -\Delta\delta &= 0 && \text{in } \Omega, \\ \delta &= 0 && \text{on } \partial\Omega, \end{aligned}$$

which is exactly the form of the Laplace equation with zero boundary values  $h(\mathbf{x}) = 0$ . From the maximum principle we infer that  $\delta(\mathbf{x}) \leq \max_{\mathbf{y} \in \partial\Omega} h(\mathbf{y}) = 0$ , and from the minimum principle that  $\delta(\mathbf{x}) \geq \min_{\mathbf{y} \in \partial\Omega} h(\mathbf{y}) = 0$ . So at every point  $\mathbf{x} \in \Omega$ , we find that  $0 \leq \delta(\mathbf{x}) \leq 0$ , which implies that  $\delta(\mathbf{x}) = 0$  everywhere. This is what we needed to prove for uniqueness.

**Problem 5 (Fourier series).** The Fourier series on  $[-L, L]$  of a function  $f(x)$  that is piecewise smooth is given by

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

where

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^L f(x) dx, \\ A_n &= \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, & B_n &= \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx. \end{aligned}$$

Calculate the Fourier series on  $[-\pi, \pi]$  of the following functions:

- a)  $f(x) = |x|$ ,
- b)  $f(x) = \cos(3x)$ ,
- c)  $f(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ -1 & \text{for } x < 0. \end{cases}$

For each of the three functions, state whether and where on the closed interval  $[-\pi, \pi]$  (i.e. including the endpoints  $x = \pm\pi$ ) the original function  $f(x)$  and its Fourier series are going to be equal. **(5 points)**

*Answer:*

a) We need to calculate the following quantities:

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{2\pi} \int_{-\pi}^0 (-x) dx + \frac{1}{2\pi} \int_0^{\pi} x dx = \frac{1}{2\pi} \pi^2 = \frac{\pi}{2}, \\ A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) \cos(nx) dx + \frac{1}{\pi} \int_0^{\pi} x \cos(nx) dx \\ &= \frac{2}{\pi n^2} (-1 + (-1)^n), \\ B_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(nx) dx = 0. \end{aligned}$$

The result for  $A_n$  is obtained by integrating by parts the integral  $\frac{1}{\pi} \int_0^\pi x \cos(nx) dx$ . The result for  $B_n$  follows immediately from the fact that the integrand is an odd function, being the product of an even and an odd function.

With this, we can write down the Fourier series of  $f(x) = |x|$  as

$$\hat{f}(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} (-1 + (-1)^n) \cos(nx).$$

In order to answer the question for which  $x$  we have that  $f(x) = \hat{f}(x)$ , recall that the points where the two might differ are exactly the points where  $f(x)$  is discontinuous, and the end points  $\pm L$  if the function was not periodic, i.e. if  $f(L) \neq f(-L)$ . Here, the function is continuous and periodic, so  $f(x) = \hat{f}(x)$  everywhere on the closed interval  $[-\pi, \pi]$ .

- b) This was a bit of a trick question: The function already has the form of a Fourier series, so we can immediately say that the coefficients are  $A_0 = A_n = B_n = 0$  with the exception of  $A_3 = 1$ . This, however, can also be deduced by remembering the orthogonality relationships

$$\int_{-L}^L \cos(nx) \cos(mx) dx = L\delta_{nm},$$

$$\int_{-L}^L \cos(nx) \sin(mx) dx = 0.$$

For the same reasons as in a), the function and its Fourier series are equal everywhere.

- c) This time we calculate for the coefficients:

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^0 (-1) dx + \frac{1}{2\pi} \int_0^{\pi} 1 dx = 0,$$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_{-\pi}^0 -\cos(nx) dx + \frac{1}{\pi} \int_0^{\pi} \cos(nx) dx = 0,$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_{-\pi}^0 -\sin(nx) dx + \frac{1}{\pi} \int_0^{\pi} \sin(nx) dx$$

$$= \frac{2}{\pi n} (1 - (-1)^n).$$

Here,  $A_0, A_n$  are zero because they are integrals over odd functions. The Fourier series of  $f(x)$  is therefore

$$\hat{f}(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n} (1 - (-1)^n) \sin(nx).$$

Not coincidentally, we could have arrived at this formula also by noting that the function  $f(x)$  here is the derivative of the one in part a) of the

problem, and since that function satisfied all the relevant criteria we could have gotten the Fourier series by term-by-term differentiation.

As for equality of  $f(x)$  and  $\hat{f}(x)$  here: the points of discontinuity are  $x = 0, \pm\pi$ , so  $f(x) = \hat{f}(x)$  everywhere except for these three points.

**I hereby certify that I have prepared my answers alone and without help by others:**

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(signature of student)

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