Problem 1 (Intersection of planes).
A hyperplane is the collection of all points \( x \in \mathbb{R}^n \) that for given \( a \in \mathbb{R}^n, \alpha \in \mathbb{R} \) satisfy the equation \( a \cdot x = \alpha \).

Consider the three planes given by
\[
\begin{bmatrix}
2 \\
0 \\
0
\end{bmatrix} \cdot x = 2, \\
\begin{bmatrix}
2 \\
4 \\
0
\end{bmatrix} \cdot x = 4, \\
\begin{bmatrix}
2 \\
4 \\
6
\end{bmatrix} \cdot x = 6.
\]

Determine whether there is one point, many points, or no points at which these three planes intersect. If there is a single, unique point, compute its coordinates.

Answer: We are obviously in \( n = 3 \) dimensions with this question. Let us write out the three equations the points \( x \) have to satisfy so that they lie in the three given planes:

\[
\begin{align*}
2x_1 + 0x_2 + 0x_3 &= 2, \\
2x_1 + 2x_2 + 0x_3 &= 4, \\
2x_1 + 4x_2 + 6x_3 &= 6.
\end{align*}
\]

The simplest way to solve this system is to see from the first equation that there is a unique \( x_1 \) that satisfies this equation, namely \( x_1 = 1 \). Knowing this, we infer from the second equation (by bringing the \( 2x_1 = 2 \) term to the right hand side) that the only possible \( x_2 \) is \( x_2 = 1/2 \). Similarly, the third equation gives us the single possible value \( x_3 = 1/3 \). Thus, we have determined that \( x = (1, 1/2, 1/3)^T \) is the unique point at which these three planes intersect.

The more convoluted way would be to see that the three equations above consitute the linear system
\[
\begin{bmatrix}
2 & 0 & 0 \\
2 & 4 & 0 \\
2 & 4 & 6
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
2 \\
4 \\
6
\end{bmatrix}.
\]

Note that the original vectors \( a_i \) constitute the rows, not columns of the matrix, as is apparent by writing out the three equations. There are again two ways to solve this: using the usual forward elimination/backward substitution steps or by realizing that this matrix is in triangular form already and that it is much simpler to solve triangulation system. The latter approach, in disguise, is of course exactly what we have done above.

In any case, one ends up with the answer that there is a single, unique point \( x = (1, 1/2, 1/3)^T \) at which these three planes intersect.

Problem 2 (Solving linear systems that are not square).
In class, we have learned what to do with linear systems where the matrix is square. Try to apply the Gaussian elimination algorithm to the following two linear systems:

\[
\begin{bmatrix}
1 & 1 \\
2 & 3 \\
3 & 4
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \\
\begin{bmatrix}
1 & 1 \\
2 & 3 \\
3 & 4
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
4
\end{bmatrix}.
\]
Describe what happens. Do these linear systems have one, many, or no solutions? (10 points)

**Answer:** If you apply the basic Gaussian elimination algorithm to these linear systems (i.e., eliminate the first column by adding multiples of the first row to the second and third), then you get the following two linear systems out of the ones given above:

\[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}.
\]

The solutions of these two linear systems are of course the same as the solutions to the original systems. We can already see here that in the left case, the last two equations of the system are the same and therefore redundant. One can then remove the last one and gets a nice linear system with a unique solution. On the other hand, the last two rows of the linear system at the right read \(0x_1 + 1x_2 = 0\) and \(0x_1 + 1x_2 = 1\); there is of course no way to find values \(x_1, x_2\) that satisfy both of these equations and, consequently, the entire linear system has no solution.

We can arrive at the same conclusion by applying a second step of Gaussian elimination to get rid of the entry at the bottom right of the matrix by adding a multiple of the second row to the last one. We then get the following:

\[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}.
\]

The interpretation here is that the last row of the system at the left contains no information since every choice of \(x\) satisfies the equation \(0x_1 + 0x_2 = 0\) and so we can drop this equation to simply obtain

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

for which the solution is \(x = (1, 0)^T\). On the other hand, the last equation of the second system was, after the second elimination step, \(0x_1 + 0x_2 = 1\) for which there is no solution. The system on the right of the question therefore has no solution.

**Problem 3 (Solving linear systems with a matrix that is a product).**

Consider the linear system

\[ C = AB \]

with

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
3 & 2 & 1
\end{pmatrix},
B = \begin{pmatrix}
3 & 2 & 1 \\
0 & 4 & 2 \\
0 & 0 & 6
\end{pmatrix}.
\]

Find \(x \in \mathbb{R}^3\) that satisfies the linear system \(Cx = \begin{pmatrix}1 \\ 2 \\ 4\end{pmatrix}\). (10 points)

**Answer:** There is an easy and a difficult way to solve this problem. The difficult way is to multiply out

\[ C = AB = \begin{pmatrix}
3 & 2 & 1 \\
6 & 8 & 4 \\
9 & 14 & 13
\end{pmatrix} \]

and then to solve the linear system \(Cx = b\) with \(b = (1, 2, 4)^T\) to obtain \(x = (1/3, -1/12, 1/6)^T\).

The smart way to solve this problem is to recognize that the two matrices are already in triangular form and remembering that solving linear systems with triangular matrices is simple. To this end, note that by
placing parentheses we can rewrite the linear system $Cx = b$ as both $(AB)x = b$ or $A(Bx) = b$. Let us call, for a moment $y = Bx$, then what we have here is a linear system $Ay = b$, i.e.,

$$
\begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
3 & 2 & 1
\end{pmatrix}
y = \begin{pmatrix} 1 \\
2 \\
4
\end{pmatrix}.
$$

This is simple: from the first equation we conclude that $y_1 = 1$. Using this, the second equation then tells us that $y_2 = 2 - 2y_1 = 0$. Finally, the last equation results in $y_3 = 4 - 3y_1 - 2y_2 = 1$.

The next step is to recover $x$ from the equation $Bx = y$, i.e.

$$
B = \begin{pmatrix} 3 & 2 & 1 \\
0 & 4 & 2 \\
0 & 0 & 6
\end{pmatrix} x = \begin{pmatrix} 1 \\
0 \\
1
\end{pmatrix}.
$$

Because the matrix is triangular, this is, again, a simple task. The last equation yields $x_3 = 1/6$. The second equation then $4x_2 = 0 - 2x_3$ or $x_2 = -1/12$. With this, the first equation gives us $3x_1 = 1 - x_3 - 2x_2$, i.e., $x_1 = 1/3$. Not coincidentally, this is exactly the same solution as we got above, albeit using a much simpler approach.

As a side note, remember that we have written $C = AB$ where $A$ is lower triangular with unit diagonal elements and $B$ is upper triangular. In other words, $C = AB$ is the LU-decomposition of the matrix $C$.

Problem 4 (Matrix inverses).
We call a matrix $A$ invertible if a matrix $B$ exists for which we have both $BA = I$ and $AB = I$. Is every $2 \times 2$ matrix invertible? If not, give an example of a matrix that is not invertible and explain why no inverse can exist for this matrix. (5 points)

Answer: No, of course not all matrices are invertible whether $2 \times 2$ or larger. A very simply matrix that is not invertible is

$$
A = \begin{pmatrix} 0 & 0 \\
0 & 0
\end{pmatrix}.
$$

Any other matrix $B$ one can conceive of would always yield $BA = AB = 0$. No other matrix could ever have entries in such a way that $BA = AB = I$.

There are, of course, many other matrices that are not invertible. Many answers to this question have given rather elaborate ones by showing that their determinants are in fact zero, using the fact that a zero determinant implies that a matrix is not invertible. An example would be

$$
A = \begin{pmatrix} 1 & 3 \\
2 & 6
\end{pmatrix}
$$

because its determinant is $\det(A) = (1)(6) - (2)(3) = 0$. That said, this is really more complicated than necessary.

Problem 5 (Cross product).
The cross product between vectors in $\mathbb{R}^3$ has two curious properties: $u \times v = -v \times u$ and $(u \times v) \cdot w = (v \times w) \cdot u = (w \times u) \cdot v$. Show how this can be used to prove that $(u \times v) \cdot u = 0$ for any two vectors $u, v \in \mathbb{R}^3$, without using what you know how the cross product is actually defined. (15 points)

Answer: The goal of this question was to show that $(u \times v) \cdot u = 0$ for any vectors $u, v$ without actually using the definition of the cross product, $u \times v$, or any of its geometric properties (such as that the cross product is proportional to the sine of the angle between vectors).
The way to do this is to first realize that the second given property implies that \((u \times w) \cdot u = (u \times u) \cdot w\). Then, consider the first given property, \(u \times v = -v \times u\). This was supposed to be true for any two vectors, \(u, v\). In particular, it therefore has to hold for \(v = u\), i.e., we have \(u \times u = -u \times u\). On the left and right we have two vectors. The statement says that the vector on the left is the negative of the vector on the right; but the two vectors and the left and right are of course the same, so this can only be the case if in fact \(u \times u = 0\). Putting this into \((u \times w) \cdot u = (u \times u) \cdot w\) shows that indeed \((u \times w) \cdot u = 0\), as claimed.

**Problem 6 (More cross products).**

Compute \(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\). (This time you are allowed to use how the cross product is actually defined.)

(5 points)

**Answer:** Using the definition of the cross product, this is a simple problem to solve. The answer is -2.

**Problem 7 (Projections).**

Let \(A \in \mathbb{R}^{n \times n}\) be an invertible matrix and \(u, v \in \mathbb{R}^n\) be two arbitrary but non-zero vectors. Then prove that the vectors

\[ p = A^{-1} \text{proj}(A^T u, Av), \quad q = \frac{\|v\|^2}{\|Av\|^2} \text{proj}(u, v) \]

are equal.

(15 points)

**Answer:** From the definition of the projection, we have that

\[ p = A^{-1} \frac{(A^T u) \cdot (Av)}{\|Av\|^2} Av, \quad q = \frac{\|v\|^2}{\|Av\|^2} \frac{u \cdot v}{\|v\|^2} = \frac{u \cdot v}{\|Av\|^2} v. \]

In the answers there were many attempts to transform the various multiplications in \(p\) to the simple enumerator of \(q\). Unfortunately, almost all of them were wrong. What is worse is that it was almost always rather trivial to see that they were wrong. Let us consider, for example, just the enumerator of \(p\). If you go back to the the definition of the projection \(\text{proj}(a, b)\) then it needs to be \(a \cdot b\). So in the case as shown above, we have that the enumerator must be \((A^T u) \cdot (Av)\) (not the parentheses) and not simply \(A^{-1} u \cdot Av\). The latter must already be wrong because the dot product is only defined between two vectors. We wouldn’t know what it means to form (reading from left to right) the product between a matrix and a vector (which results in a vector) and then the dot product with a matrix. Now, consider for a moment that we would want to interpret this in the following way: \((A^T)^{-1} [u \cdot (Av)]\), which at least is possible with the dot product. Then the result would be a matrix \((A^T)^{-1}\) multiplied by the scalar that results from forming a dot product. So the result would be a matrix. But in the original definition of the projection, the enumerator was \(a \cdot b\), i.e., a scalar. In other words, the interpretation just shown can not possibly be correct: it really must be \((A^T)^{-1} u \cdot (Av)\), with the parentheses interpreted as shown.

So far, there has not been any particular insight other than things we should obviously not do because they simply make no sense. Following this reasoning, let us consider a bit more what we have with the vector

\[ p = A^{-1} \frac{(A^T^{-1} u) \cdot (Av)}{\|Av\|^2} Av. \]

Here, the fraction is a ratio between two scalars, so it’s a scalar. In other words, \(p\) is the product between a matrix, a scalar, a matrix and a vector. We know that we can commute the multiplication with scalars (unlike the multiplication between matrices!). So this would give us

\[ p = \frac{(A^T)^{-1} u \cdot (Av)}{\|Av\|^2} A^{-1} Av = \frac{(A^T)^{-1} u \cdot (Av)}{\|Av\|^2} v. \]
OK, that’s already closer to what we need: the vector on the right is already correct, and so is the denominator of the fraction. Now let’s look at that enumerator of the fraction again: \((A^T)^{-1}u \cdot (Av)\). To get this to the desired form \(u \cdot v\) there are only two and a half real insights you need to remember:

- The dot product between two vectors is identical to the matrix product between the transpose of the first vector when interpreted as a matrix times the second vector when interpreted as a matrix. In other words, we have \(a \cdot b = a^Tb\). Applied to the current situation, this yields:
  \[((A^T)^{-1}u) \cdot (Av) = ((A^T)^{-1}u)^T(Av)\].

- When you have matrices \(D, E\) (or vectors considered as a matrix with a single column) then you have \((DE)^T = E^TD^T\). Again applied to the current situation we then have
  \[((A^T)^{-1}u) \cdot (Av) = ((A^T)^{-1}u)^T(Av) = u^T((A^T)^{-1})^T(Av)\].

- Finally, remember that \((A^T)^{-1} = (A^{-1})^T\). This means that \((A^T)^{-1}v = A^{-1}v\). In other words, using the associativity of matrix matrix products, we get that
  \[((A^T)^{-1}u) \cdot (Av) = ((A^T)^{-1}u)^T(Av) = u^T(A^{-1})Av = u^TA^{-1}v = u^Tv = u \cdot v\).

Putting it all back together then yields the claimed result.

**Problem 8 (Linear combinations).**

Let \(V\) be the vector space \(P_3(t)\) over \(\mathbb{R}\) (i.e., of polynomials of degree at most 3 with coefficients in \(\mathbb{R}\)). Consider the functions \(v(t) = 1 - t^2 \in V\) and \(u_1(t) = 1 - t, u_2(t) = 1 + t, u_3(t) = t + 4t^2 \in V\). Define what it means for \(v\) to be a linear combination of \(u_1, u_2, u_3\). Is \(v\) a linear combination of \(u_1, u_2, u_3\)? If so, explain why.

**Answer:** For \(v(t)\) to be a linear combination of \(u_1(t), u_2(t), u_3(t)\) means that there must exist constants \(a_1, a_2, a_3 \in \mathbb{R}\) so that \(v(t) = a_1u_1(t) + a_2u_2(t) + a_3u_3(t)\). Writing this out yields the following equation:

\[
1 - t^2 = a_1(1 - t) + a_2(1 + t) + a_3(t + 4t^2)
\]

\[
= (a_1 + a_2) + (-a_1 + a_2 + a_3)t + 4a_3t^2
\]

If the left and right hand sides are to be equal for all possible \(t\), then this implies that

\[
a_1 + a_2 = 1,
-a_1 + a_2 + a_3 = 0, 
4a_3 = 1.
\]

This is a linear system that we can solve, either directly, or by rewriting it as

\[
\begin{pmatrix}
1 & 1 & 0 \\
-1 & 1 & 1 \\
0 & 0 & 4
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}.
\]

A solution of to this linear system exists and is \(a = (5/8, 3/8, 1/4)^T\). In other words, this implies that

\[
v(t) = \frac{5}{8}u_1(t) + \frac{3}{8}u_2(t) + \frac{1}{4}u_3(t).
\]

In other words, \(v\) is indeed a linear combination of the \(u_i\).

I hereby certify that I have prepared my answers alone and without help by others:
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