Problem 1 (Norms on infinite dimensional spaces). To show that on finite intervals, i.e., where \(|b-a| < \infty\), these two are in fact norms, let us note that the linearity and the fact that the norm is only zero if \(f(x)\) is zero everywhere, are easy to show.

The only tricky part is the triangle inequality. Let us show this first for the supremum norm. Consider two functions \(f(x)\) and \(g(x)\). Then

\[
\|f + g\| = \max_{x \in [a,b]} |f(x) + g(x)|
\]

\[
\leq \max_{x \in [a,b]} (|f(x)| + |g(x)|) \leq \max_{x \in [a,b]} |f(x)| + \max_{x \in [a,b]} |g(x)|
\]

\[
= \|f\| + \|g\|.
\]

In other words, the maximum of the function \((f + g)(x)\) is less than or equal than the sum of maxima. This is easy enough to understand by just visualizing the sum of two functions.

For the \(L_1\) norm, we prove the triangle inequality as follow:

\[
\|f + g\|' = \int_a^b |f(x) + g(x)|
\]

\[
\leq \int_a^b (|f(x)| + |g(x)|) = \int_a^b |f(x)| + \int_a^b |g(x)|
\]

\[
= \|f\|' + \|g\|'.
\]

This proves that both of the proposed norms indeed satisfy the three conditions.

To show that the norms are not equivalent, the question already laid out the way to a proof. In essence, we need to find a sequence of functions \(\{f_n\}\) for which, for example, the supremum stays constant but the integral goes to zero, or to infinity.

For bounded intervals, it turns out that only the former is possible. Think, for example, of the following set of functions:

\[
f_n(x) = e^{-n|x|}.
\]

(Let us pretend for a moment that we were on the interval \([a, b] = [-1, 1]\); if we were on a different interval, we would just need to shift the functions \(f_n\) to the left or right.) These functions all have maximum equal to one, i.e., \(\|f_n\| = |f(0)| = 1\). But they are decaying exponentials for \(x > 0\) and their mirror image to the left that become narrower and narrower as \(n \to \infty\), i.e., the
area under the curve becomes smaller and smaller. Indeed, it is easy to verify that

\[ \|f_n\|' = \int_{-1}^{1} f_n = 2 \int_{0}^{1} e^{-nx} = \frac{2}{n} (1 - e^{-n}) \to 0. \]

In other words, for this sequence of functions \( \|f_n\|/\|f_n\|' \to \infty \) and the one half of the norm equivalence is already not satisfied.

For finite intervals, the other half holds, however: if the maximum of a function \( f \) is, say, \( M \) on the interval, i.e., \( |f(x)| \leq M \) for all \( x \in [a,b] \), then clearly we have that

\[ \|f\|' = \int_{a}^{b} |f(x)| \leq \int_{a}^{b} M = M(b - a) = (b - a)\|f\|, \]

which shows this half of the norm equivalence. In other words, if the function’s value is bounded by \( M \), then the area under its curve can be no larger than the width of the interval \( b - a \) times the height \( M \) of the function.

This latter argument can only work, however, if \( b - a \) is indeed finite. For infinite intervals, the right hand side of the inequality immediately above blows up. We can show this again for the same type of function above. Consider now

\[ g_n(x) = e^{-|x|}/n, \]

and let us for simplicity consider the interval \(( -\infty, \infty )\). Again, these functions all have maximum equal to one, i.e., \( \|g_n\| = \|f(0)\| = 1 \). They are also decaying exponentials for \( x > 0 \) and their mirror image to the left, but if I now make \( n \) larger and larger they become flatter and flatter, getting closer to the function that is constant one everywhere. Indeed, the area under the curve is

\[ \|g_n\|' = \int_{-\infty}^{\infty} g_n = 2 \int_{0}^{\infty} e^{-x/n} = 2n \rightarrow \infty. \]

That is, for this sequence of functions, \( \|g_n\|'/\|f\| \to \infty \). This violates the other half of the norm equivalence. The first half is still violated, using the same functions \( f_n \) as before. In other words, neither direction of the norm equivalence works on infinite intervals.

**Problem 2 (Jacobi iteration).** The following program (shown here with \( \omega = 1 \)) implements the damped Jacobi iteration for the simple case of the \( 2 \times 2 \) matrix of the problem:

```cpp
#include <iostream>

int main ()
{
    double M[2][2];
    double x[2], tmp[2], b[2];
```
M[0][0] = M[0][1] = 1;
M[1][0] = -2;
M[1][1] = 2;
b[0] = b[1] = 1;

double omega = 1;

for (unsigned int k=0; k<20; ++k)
{
    // compute residual
    tmp[0] = b[0] - (M[0][0]*x[0] + M[0][1]*x[1]);
    tmp[1] = b[1] - (M[1][0]*x[0] + M[1][1]*x[1]);

    // apply D^{-1}
    tmp[0] /= M[0][0];
    tmp[1] /= M[1][1];

    // scale by omega
    tmp[0] *= omega;
    tmp[1] *= omega;

    // add it to the previous iterate
    x[0] += tmp[0];
    x[1] += tmp[1];

    // print the current iterate
    std::cout << x[0] << ' ' << x[1] << std::endl;
}

If you run this program, you will get output of the following kind:

1 0.5
0.5 1.5
-0.5 1
0 0
1 0.5
0.5 1.5
...

In other words, for ω = 1 the iterates cycle every four iterations and will not converge.

One can play with the value of ω in the program. For example, choosing ω = 0.5 yields this output:

0.5 0.25
0.625 0.625
Looking at the values, this suggests that the solution may be \( x = (0.25, 0.75)^T \) which is of course indeed true. If we run some more iterations, we can indeed confirm that the damped Jacobi iteration converges to these values if \( \omega = 0.5 \). Based on this, we can try other values of the damping parameter. In some cases, we may have to run many more than just the 20 iterations shown in the program to see whether iterations converge for a particular value of \( \omega \), but if you do, you will find that every value \( 0 < \omega < 1 \) actually leads to convergence, albeit very slow convergence for values \( \omega \) close to zero or one.

To prove this, recall that the general theory of fixed point iterations of the form

\[
x^{(k+1)} = x^{(k)} + B(b - Ax^{(k)})
\]

yielded the result that the iteration converges if \( \|I - BA\| \leq \delta \) for some \( \delta < 1 \), in any conveniently chosen norm. In class, we had shown that the Jacobi iteration converges for strictly diagonally dominant matrices if one chooses the \( l_\infty \) norm. Here, we will need to choose a different norm, but first let us remark that for the damped Jacobi iteration, we have \( B = \omega [\text{diag}(A)]^{-1} \). With this, the matrix for which we need to show something is

\[
I - BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \omega \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \omega \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 - \omega & -\omega \\ \omega & 1 - \omega \end{pmatrix}
\]

For non-negative \( \omega \), this matrix has \( l_1 \) and \( l_\infty \) norm equal to one, so this norm is not useful to us in showing convergence. However, we may try the \( l_2 \) norm. To this end, we need the eigenvalues of this matrix. A quick computation shows that they are \( \lambda_1 = 1 - \omega \pm \omega i \). This means that \( |\lambda_1| = |\lambda_2| = \sqrt{(1 - \omega)^2 + \omega^2} \). In other words, we have just shown that

\[
\|I - BA\|_{l_2} = \sqrt{(1 - \omega)^2 + \omega^2}.
\]
This is indeed smaller than one for any choice of $0 < \omega < 1$, but it is equal to one for $\omega = 0$ and $\omega = 1$. What we have then just shown is that the iteration converges (for this particular matrix) for any choice $0 < \omega < 1$.

Furthermore, because $\|I - BA\|_2 \to 1$ as $\omega \to 0$ or $\omega \to 1$, we should not be surprised that the iteration becomes arbitrarily slow if we choose a damping parameter close to zero or one. Note, however, that I said “we should not be surprised” rather than “we have shown that the iteration becomes arbitrarily slow”: just because $\|I - BA\|_2$ is close to or equal to one in the $l_2$ norm does not imply that it is also close to or equal to one in another norm; if it were smaller in another norm, this would imply quick convergence, even if we can’t show this using the $l_2$ norm. Our inability to prove convergence for any value of $\omega$ by looking at the $l_1$ and $l_\infty$ norms should have taught us that already!