

1.3 Algebraic Expressions

A **polynomial** is an expression of the form:

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

The numbers a_1, a_2, \dots, a_n are called **coefficients**.

Each of the separate parts, such as $a_3 x^3$ or $a_1 x$, is called a **term** of the polynomial.

If there is only one term, it is called a monomial. For two terms, it is called a binomial. For three terms, a trinomial.

The **degree** of the polynomial is n : the highest power of x .

To add (or subtract) polynomials, we combine like terms (those that have the same variables raised to the same powers).

Example: Find the difference $(x^3 - 4x^2 + 6x + 12) - (2x^3 + x^2 - 4x - 6)$

To multiply polynomials, we use the distributive laws. In particular, to multiply two binomials, we use the FOIL method.

Example: Multiply $(x - 7)(2x + 3)(x - 1)$

Special Product and Factoring Formulas

1. Difference of Squares

$$(A + B)(A - B) = A^2 - B^2$$

2. Perfect Squares

$$(a) \quad (A + B)^2 = A^2 + 2AB + B^2$$

$$(b) \quad (A - B)^2 = A^2 - 2AB + B^2$$

3. Cubing a Sum or Difference

$$(a) \quad (A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

$$(b) \quad (A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$$

4. Sum or Difference of Cubes

$$(a) \quad A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

$$(b) \quad A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

Examples: Expand the following.

- $(2\sqrt{x} + 3)(2\sqrt{x} - 3)$

- $(x^3 - 4)^2$

- $(x^2 - 2)^3$

Factoring Steps

1. Factor out all common factors.
2. See if you can use a special factoring formula.
3. See if you can factor by grouping.
4. Use trial and error.

Examples: Factor the following.

- $8x^2 - 24x + 18$

- $36x^2 - 25$

- $x^3 - 27$

- $2x^2 + 11x - 21$

- $x^3 + 3x^2 - x - 3$

- $(a^2 + 2a)^2 - 2(a^2 + 2a) - 3$

To factor expressions with rational exponents, first factor out, if possible, the *smallest* power of x .

- Factor $x^{-3/2} + 2x^{-1/2} + x^{1/2}$

Its easy to check if you factored correctly. Just multiply back out to double check.

1.4 Rational Expressions

A **rational expression** is a fractional expression where both the numerator and denominator are polynomials.

The **domain** of any algebraic expression is the set of values that the variable can be.

So far, we have two things to look for to determine the domain:

- The denominator can't be zero. If a value of x makes the denominator zero, we must exclude it from the domain.
- For even powered roots (square roots, fourth roots, etc), whatever is under the radical must be ≥ 0 .

Examples: Find the domains of the following rational expressions.

1. $\frac{x}{2x - 3}$

2. $\frac{\sqrt{x}}{x^2 - 9}$

3. $\frac{x^2 + 3x}{\sqrt{x + 3}}$

Working with rational expressions is just like working with fractions.

To multiply rational expressions, factor the numerator and denominator, multiply, and then simplify by cancelling common factors in the numerator and denominator.

To divide rational expressions, multiply by the reciprocal.

To add or subtract rational expressions, you **MUST** have a **COMMON DENOMINATOR** (just like with fractions).

Examples

- $\frac{x^2 - 1}{x^2 - 7x + 12} \cdot \frac{x^2 - x - 6}{x + 1}$

$$\bullet \frac{4y^2 - 9}{2y^2 + 9y - 18} \div \frac{2y^2 + y - 3}{y^2 + 5y - 6}$$

$$\bullet \frac{x}{x^2 + x - 2} - \frac{2}{x^2 - 5x + 4}$$

Compound fractions are fractional expressions where the numerator and/or denominator are themselves fractional expressions.

Examples:

$$\bullet \frac{\frac{x}{x+y}}{\frac{x}{y} + \frac{y}{x}}$$

$$\bullet \frac{3(1+x)^{1/3} - x(1+x)^{-2/3}}{(1+x)^{2/3}}$$

Rationalizing the Denominator or Numerator

As before, rationalizing means to get rid of any radicals. We do this by multiplying by the **conjugate**. The conjugate is usually found by just changing the sign of the second term.

For example, the conjugate of $A + B\sqrt{C}$ is $A - B\sqrt{C}$. What's the point? When you multiply these two expressions together you get an expression that has NO radicals: $(A + B\sqrt{C})(A - B\sqrt{C}) = A^2 - B^2C$

Examples

- What are the conjugates of:

$$2 - 3\sqrt{7}$$

$$\sqrt{x-3} - 9$$

$$\sqrt{x} + \sqrt{y}$$

- Rationalize the denominator of $\frac{x}{4 - \sqrt{x}}$.

- Rationalize the numerator of $\frac{\sqrt{3+x} + 2}{3}$.

1.5 Equations

Linear equations can always be written in the form $ax + b = 0$.

To solve linear equations, use basic algebra, but remember what you do to one side, you **MUST** do to the other side.

Example: Solve the following linear equation for the variable x .

$$\frac{ax + b}{cx + d} = 2$$

Quadratic equations can be written in the form $ax^2 + bx + c = 0$ with $a \neq 0$.

To solve a quadratic equation by factoring, we use the **Zero-Product Property**, which tells us that if $AB = 0$, then $A = 0$ or $B = 0$. Note: This does NOT work if the right hand side is not 0. So, if $AB = 2$, it does NOT mean that $A = 2$ or $B = 2$.

Example: Solve this equation by factoring: $2x^2 - x = 3$

To solve quadratics that are written as perfect squares, just take square roots of both sides.

Example: Solve $(x - 5)^2 = 17$

If an equation does not easily factor, we can use the method of **completing the square**. The idea is to get the equation in the form of the example above, where the equation has a perfect square in it.

To make $x^2 + bx$ a perfect square, add $(\frac{b}{2})^2$.

Example: Solve this equation by completing the square: $3x^2 - 6x - 1 = 0$

What happens if a quadratic is not factorable or you can't remember how to factor it? Use **THE QUADRATIC FORMULA**. It **ALWAYS** works.

The roots (or solutions) of the quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$, are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solve the equation $3x^2 - 6x - 1 = 0$ by using the Quadratic Formula.

The **discriminant** of a quadratic equation is $D = b^2 - 4ac$. It is the part of the quadratic formula that is underneath the square root. The discriminant tells us how many real solutions the equation has. There are 3 cases.

1. $D > 0$
2. $D = 0$
3. $D < 0$

Example: How many real solutions does $4x^2 - 5x + 2 = 0$ have?

An object thrown up at an initial speed of v_0 ft/s will reach a height of h feet after t seconds based on the formula

$$h = -16t^2 + v_0t$$

Suppose I throw a ball up with a speed of 48 ft/s.

a) When will the ball fall back down to me?

b) When will it reach a height of 32 ft.?

c) Will it reach a height of 64 ft.?

Other types of equations

- Equations with Fractional Expressions. Multiply both sides by the LCD and then solve.

$$\frac{x+5}{x-2} = \frac{5}{x+2} + \frac{28}{x^2-4}$$

However, we must then check for **extraneous solutions**, which are potential solutions that do not actually satisfy the original equation. Whenever we multiply an equation by something with the variable, or whenever we square both sides of an equation, we must check for extraneous solutions. In general, it is a good idea to check your answers.

- Equations with Radicals: Isolate the square root on one side of the equation and then square both sides. Remember to check for extraneous solutions.

$$\sqrt{5-x} + 1 = x - 2$$

- Equations of Quadratic Type: The idea is to get the equation in the form of a quadratic, which we know how to solve. After solving, again check for extraneous solutions.

$$x^4 - 5x^2 + 4 = 0$$

$$x^{4/3} - 5x^{2/3} + 6 = 0$$

- Absolute Value Equations: Isolate the absolute value and separate into cases.

$$|4x + 7| = 2$$