Chapter 8: Vectors

8.4 Vectors

A number by itself is called a **scalar**.

A **vector** is a line segment with a direction assigned to it. Vectors consist of both magnitude and direction.

Vectors are denoted by boldface letters: **u**. However, since you cannot write boldface, we also denote vectors by $\vec{u}$.

The **magnitude** of a vector is its length, denoted $|\mathbf{u}|$.

If a vector starts at a point A and ends at a point B, we say A is the **initial point** and B is the **terminal point**. We would denote this by $\mathbf{u}=\overrightarrow{AB}$.

Multiplying a vector by a scalar (number) changes the magnitude of the vector by this factor. A negative scalar changes the magnitude and also reverses the direction.

Two vectors are equal if they have the same magnitude and direction. So, it doesn't matter *where* they are as long as they have the same magnitude and direction (the same displacement).

If $\mathbf{u}=\overrightarrow{AB}$ and $\mathbf{v}=\overrightarrow{BC}$, then $\mathbf{u}+\mathbf{v}$ is the vector $\overrightarrow{AC}$. This, as well as $\mathbf{u}-\mathbf{v}$, can be shown graphically as follows.

Usually, instead of writing a vector as **v**=$\overrightarrow{AB}$, it is more descriptive to write the vector in terms of its horizontal and vertical components.

We write **v**=$< a, b >$, where a is the horizontal component and b is the vertical component.

If **v** is a vector with initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$, then we write

$$\mathbf{v} =< x_2 - x_1, y_2 - y_1 >$$

The zero vector, **0**, is the vector $< 0, 0 >$.

For two vectors to be equal, they must have the same vertical and horizontal components.

Example: Find the component form of the vector **u** with initial point $(-1, 3)$ and terminal point $(-6, 1)$.

The magnitude (or length) of a vector **v**=$< a, b >$ is

$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

Why?

Find the magnitude of the vector **u** found in the above example.

To add or subtract vectors, we just add or subtract their corresponding horizontal and vertical components. To multiply a vector by a scalar, we multiply both the horizontal and vertical components by this scalar.

Example: Let **u**=$< -2, 5 >$ and **v**=$< 2, -8 >$. Calculate -5**v**, **u**+**v**, and $-$**u**+3**v**.

A vector with length 1 is called a **unit vector**.
If we have a vector **v** and we want a unit vector pointing in the direction of **v**, all we do is calculate $\frac{\mathbf{v}}{|\mathbf{v}|}$.
Example: If $\mathbf{v} =< -2, 3 >$, find a unit vector in the direction of **v**.

There are two special unit vectors that we use all the time:

$$\mathbf{i} =< 1, 0 > \qquad \mathbf{j} =< 0, 1 >$$

i is the unit vector in the horizontal (x) direction, and **j** is the unit vector in the vertical (y) direction.

EVERY vector $\mathbf{v}=< a, b >$ can be written in terms of these unit vectors by

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j}$$

Example: Write the vectors $\mathbf{u}=< -6, 6 >$ and $\mathbf{v}=< 2, 1 >$ in terms of **i** and **j**. Then, find $2\mathbf{u} - 4\mathbf{v}$ (in terms of **i** and **j**), and calculate $|2\mathbf{u} - 4\mathbf{v}|$.

The direction of a vector is the positive angle θ formed by the positive x-axis and the vector.

Suppose that we have a vector **v** where we know the magnitude $|\mathbf{v}|$ and direction θ of the vector. Then $\mathbf{v}=< a, b >= a\mathbf{i} + b\mathbf{j}$, where

$$a = |\mathbf{v}| \cos\theta \text{ and } b = |\mathbf{v}| \sin\theta$$

Example: Suppose I have a vector **v** where $|\mathbf{v}|$=4 and $\theta = 60°$. Find the horizontal and vertical components and write the vector in terms of **i** and **j**.

Conversely, if we are given the components of a vector, $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, we can find the direction of the vector by knowing which quadrant the terminal point is in and using the fact that $\tan\theta = \frac{b}{a}$.

Example: Find the magnitude and direction of the vector $\mathbf{v} = -\mathbf{i} + \sqrt{3}\mathbf{j}$.

The velocity of an object can be modeled by a vector, where the direction of the vector is the direction of motion, and the magnitude of the vector is the speed.

Applied Examples

1. A salmon is swimming N 60° E at 5 mi/h relative to the water. The prevailing ocean current flows due east at 3 mi/h. Find the true velocity of the fish as a vector.

2. A jet is flying through a wind that is blowing with a speed of 55 mi/h in the direction N 30° W. The jet has a speed of 765 mi/h relative to the air, and the pilot heads the jet in the direction N 45° E.

   (a) Express the velocity of the wind and the velocity of the jet relative to the air as vectors in component form.

   (b) Find the true velocity of the jet as a vector.

   (c) Find the true speed and direction of the jet.

8.5 The Dot Product

If $\mathbf{u}=< x_1, y_1 >$ and $\mathbf{v}=< x_2, y_2 >$, then the **dot product** of these vectors, denoted by $\mathbf{u}\cdot\mathbf{v}$, is defined to be

$$\mathbf{u}\cdot\mathbf{v} = x_1x_2 + y_1y_2$$

Note that the dot product is NOT a vector, it is a scalar.

Example: If $\mathbf{u} =< 3, -2 >$ and $\mathbf{v} =< 1, 2 >$, find $\mathbf{u}\cdot\mathbf{v}$.

Properties of the Dot Product

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
2. $(a\mathbf{u})\cdot\mathbf{v} = a(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(a\mathbf{v})$
3. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$
4. $|\mathbf{u}|^2 = \mathbf{u}\cdot\mathbf{u}$

Show that the last property is true for a general vector $\mathbf{u} =< a, b >$.

The dot product is helpful in determining the angle between two nonzero vectors.

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

Example: If $\mathbf{u}$ has length 3, $\mathbf{v}$ has length 5, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is 60°, find $\mathbf{u}\cdot\mathbf{v}$.

This leads immediately to the fact that if θ is the angle between two nonzero vectors, then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

Example: If $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$, find the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are **perpendicular** or **orthogonal** if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

Why? Because two vectors are orthogonal when the angle between them is $\frac{\pi}{2}$ and $\cos\frac{\pi}{2} = 0$.

Example: Determine whether the vectors $\mathbf{u} =< -2, 6 >$ and $\mathbf{v} =< 4, 2 >$ are orthogonal.