Math 150 Week in Review 5 Problem Set

1. Determine the end behavior of the following polynomials.

   (a) $P(x) = 5x^8 + 6x^7 - 4x - 9$

   Even degree; Positive leading coefficient

   As $x \to \infty$, $y \to \infty$

   As $x \to -\infty$, $y \to \infty$

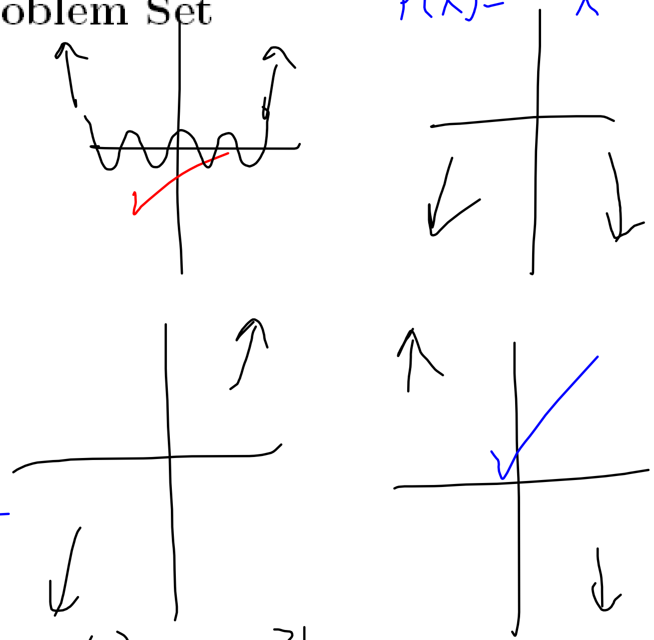
   (b) $P(x) = -10x^{11} - 5x^6 + 5x^2 - 2$

   Odd degree; Negative leading coefficient

   As $x \to \infty$, $y \to -\infty$

   As $x \to -\infty$, $y \to \infty$

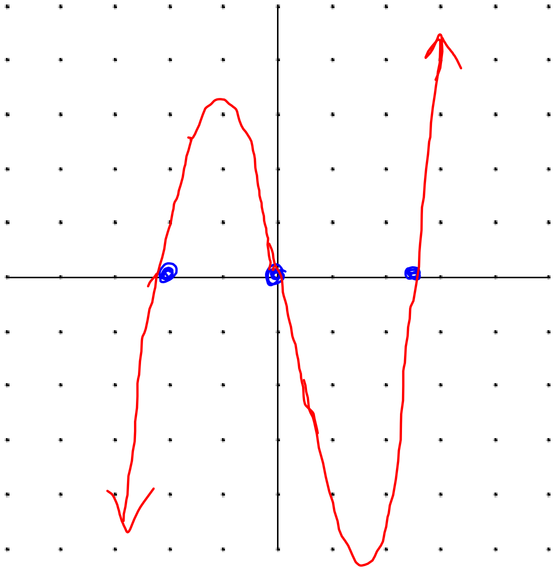
$P(x) = -x^{82}$



$P(x) = 20x^{71}$

2. Sketch graphs for the following polynomials.

(a) $P(x) = 2x^3 - x^2 - 10x = x(2x^2 - x - 10) = x(2x-5)(x+2)$



Zeros: (x-int)

$x = 0, \frac{5}{2}, -2$

$(0,0)$ $(\frac{5}{2}, 0)$ $(-2, 0)$

y-int: $P(0) = 0$: $(0,0)$

End behavior:

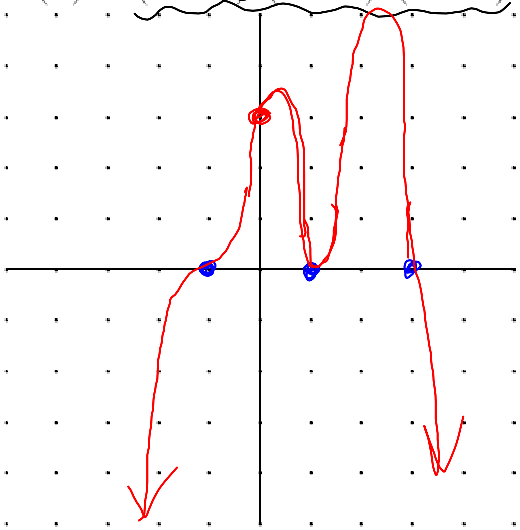
Odd degree; Positive lead. coeff.

As $x \to \infty$, $y \to \infty$

$x \to -\infty$, $y \to -\infty$

Each zero has odd multiplicity $\to$ $P(x)$ crosses x-axis at each zero.

(b) $P(x) = (x-1)^2(x+1)^3(3-x)$



zeros: 1, −1, 3

x-int: (1, 0), (−1, 0), (3, 0)

y-int: $P(0) = (0-1)^2(0+1)^3(3-0)$

$= (-1)^2(1)^3(3) = 3$

(0, 3)

End behavior:

Leading Term: $x^2 \cdot x^3(-x) = -x^6$

Even degree, Negative leading coef.

As $x \to \infty$, $y \to -\infty$

As $x \to -\infty$, $y \to -\infty$

−1 is a zero of multiplicity 3.

→ crosses x-axis: like x^3

1 is a zero of multiplicity 2

→ touches only: like x^2.

3 is a zero of multiplicity 1

→ crosses x-axis: like x.

3. Use long division to find the quotient and remainder of the following.

(a) $\dfrac{2x^5 + 4x^3 - 6x + 3}{x^2 - 2x + 1}$

$\dfrac{2x^5}{x^2} = 2x^3$

$$
\begin{array}{r}
2x^3 + 4x^2 + 10x + 16 \\
x^2 - 2x + 1 \enclose{longdiv}{2x^5 + 0x^4 + 4x^3 + 0x^2 - 6x + 3} \\
-(2x^5 - 4x^4 + 2x^3) \\
\hline
4x^4 + 2x^3 + 0x^2 - 6x + 3 \\
-(4x^4 - 8x^3 + 4x^2) \\
\hline
10x^3 - 4x^2 - 6x + 3 \\
-(10x^3 - 20x^2 + 10x) \\
\hline
16x^2 - 16x + 3 \\
-(16x^2 - 32x + 16) \\
\hline
16x - 13
\end{array}
$$

$$\boxed{\begin{aligned} Q(x) &= 2x^3 + 4x^2 + 10x + 16 \\ R(x) &= 16x - 13 \end{aligned}}$$

(b) $\dfrac{3x^3 - 3x^2 + x - 4}{-2x + 3}$

$$
\begin{array}{r|l}
 & -\frac{3}{2}x^2 - \frac{3}{4}x - \frac{13}{8} \\
\hline
-2x+3 & 3x^3 - 3x^2 + x - 4 \\
 & -\left(3x^3 - \frac{9}{2}x^2\right) \\
\hline
 & \frac{3}{2}x^2 + x - 4 \\
 & -\left(\frac{3}{2}x^2 - \frac{9}{4}x\right) \\
\hline
 & \frac{13}{4}x - 4 \\
 & -\left(\frac{13}{4}x - \frac{39}{8}\right) \\
\hline
 & \frac{7}{8}
\end{array}
$$

$\frac{3x^3}{-2x} = -\frac{3}{2}x^2$

$-3 + \frac{9}{2}$

$= \frac{-6}{2} + \frac{9}{2} = \frac{3}{2}$

$\frac{\frac{3}{2}x^2}{-2x} = -\frac{3}{4}x$

$1 + \frac{9}{4} = \frac{13}{4}$

$\frac{\frac{13}{4}x}{-2x} = -\frac{13}{8}$

$-4 + \frac{39}{8}$

$\frac{-32}{8} + \frac{39}{8} = \frac{7}{8}$

$$\boxed{\begin{aligned} Q(x) &= -\frac{3}{2}x^2 - \frac{3}{4}x - \frac{13}{8} \\ R(x) &= \frac{7}{8} \end{aligned}}$$

4. Find the polynomial of degree 4 with zeros 1, −2, and 3, where 1 is a zero of multiplicity 2 and the constant term is 18.

$$P(x) = a(x-1)^2(x+2)(x-3)$$

$$P(x) = a(x^2-2x+1)(x^2-x-6)$$

$$= a(x^4 - x^3 - 6x^2 - 2x^3 + 2x^2 + 12x + x^2 - x - 6)$$

$$= a(x^4 - 3x^3 - 3x^2 + 11x - 6)$$

Let $a = -3$

$$P(x) = -3(x^4 - 3x^3 - 3x^2 + 11x - 6)$$

$$\boxed{P(x) = -3x^4 + 9x^3 + 9x^2 - 33x + 18}$$

5. Evaluate the following expressions and write in standard form.

(a) $i^{30} = i^{28} \cdot i^2 = 1 \cdot i^2 = \boxed{-1}$

$$\left.\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}\right\}$$

$$\begin{aligned} i^5 &= i \\ i^6 &= -1 \end{aligned}$$

$\sqrt{-1} = i$

(b) $(5 - \sqrt{-9})(-2 + \sqrt{-12})$

$(5 - i\sqrt{9})(-2 + i\sqrt{12})$

$(5 - 3i)(-2 + 2i\sqrt{3})$

$-10 + 10i\sqrt{3} + 6i - 6\overset{=-1}{i^2}\sqrt{3}$

$-10 + 10i\sqrt{3} + 6i + 6\sqrt{3}$

$= \boxed{(-10 + 6\sqrt{3}) + (10\sqrt{3} + 6)i}$

(c) $\dfrac{7-4i}{2-5i} \cdot \dfrac{2+5i}{2+5i} = \dfrac{(7-4i)(2+5i)}{2^2 + 5^2}$

$= \dfrac{14 + 35i - 8i - 20\overset{=-1}{i^2}}{29}$

$= \dfrac{34 + 27i}{29} = \boxed{\dfrac{34}{29} + \dfrac{27}{29}i}$

$z = a + bi$

$\bar{z} = a - bi$

$z\bar{z} = a^2 + b^2$

6. Solve the equation $3x^2 - 2x = -1$.

$$3x^2 - 2x + 1 = 0$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(1)}}{2(3)}$$

$$\sqrt{-8} = i\sqrt{8}$$
$$= 2i\sqrt{2}$$

$$= \frac{2 \pm \sqrt{-8}}{6} = \frac{\cancel{2} \pm \cancel{2}i\sqrt{2}}{\cancel{6}\,3}$$

$$= \boxed{\frac{1 \pm i\sqrt{2}}{3} = \frac{1}{3} \pm \frac{\sqrt{2}}{3}i}$$

7. Find all intercepts, vertical or horizontal asymptotes, and holes for the following rational functions.

(a) $r(x) = \dfrac{(3x^2-12)(2x-1)}{(4x^2+4x-3)(x+2)}$ $= \dfrac{3(x^2-4)(2x-1)}{(2x+3)(2x-1)(x+2)} = \dfrac{3(x-2)\cancel{(x+2)}\cancel{(2x-1)}}{(2x+3)\cancel{(2x-1)}\cancel{(x+2)}}$

$= \dfrac{3(x-2)}{2x+3}, \quad x \neq \frac{1}{2}, -2$

x-int: $x=2$; $\underline{(2,0)}$

y-int: $r(0) = \dfrac{3(0-2)}{2(0)+3} = \dfrac{-6}{3} = -2$

$\underline{\underline{(0,-2)}}$

Holes at $x = 1/2, -2$

VA: $x = -3/2$

HA: Deg Num $= 1$

Deg Den $= 1$

$\boxed{y = \dfrac{3}{2}}$

(b) $r(x) = \dfrac{x^3 + 2x^2 - 8x}{2x^2 - 8x - 10} = \dfrac{x(x^2+2x-8)}{2(x^2-4x-5)} = \dfrac{x(x+4)(x-2)}{2(x-5)(x+1)}$ ✓

x-int: $x = 0, -4, 2$

$(0,0)$, $(-4,0)$, $(2,0)$

y-int: $r(0) = \dfrac{0}{-10} = 0$

$(0,0)$

No holes since no factors cancel.

VA: $x = 5$, $x = -1$

HA: Deg Num = 3

Deg Den = 2

Since Deg Num > Deg Den

No horizontal asymptote

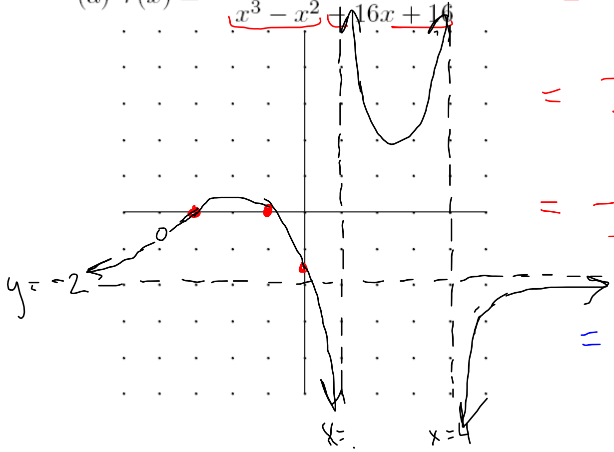
8. Sketch graphs for the following rational functions.

(a) $r(x) = \dfrac{-2(x^2+5x+4)(x+3)}{x^3-x^2-16x+16}$ $= \dfrac{-2(x+4)(x+1)(x+3)}{x^2(x-1)-16(x-1)}$

$= \dfrac{-2(x+4)(x+1)(x+3)}{(x-1)(x^2-16)}$

$= \dfrac{-2\cancel{(x+4)}(x+1)(x+3)}{(x-1)(x-4)\cancel{(x+4)}}$

$= \dfrac{-2(x+1)(x+3)}{(x-1)(x-4)}$, $x \neq -4$



Hole at $x=-4$

VA: $x=1,\ x=4$

HA: $r(x) = \dfrac{-2x^2+\dots}{x^2+\dots}$

Since Deg Num $= 2 =$ Deg Den, then $y = \dfrac{-2}{1} = -2$

$y=-2$

	-3		-1		1		4		
+		+		−		−		−	$-2(x+1)$
−		+		+		+		+	$x+3$
−		−		−		−		+	$x-4$
−		−		−		+		+	$x-1$
−		+		−		+		−	
below x-axis		above		below		above		below	

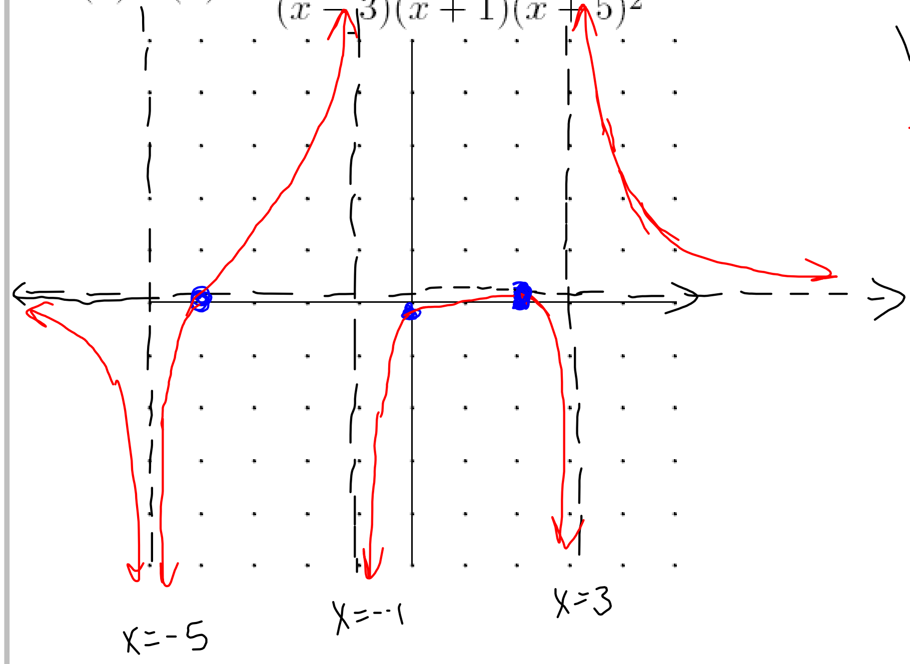
x-int: $x=-1,-3$

$(-1,0)$, $(-3,0)$

y-int: $r(0) = \dfrac{-2(1)(3)}{(-1)(-4)} = \dfrac{-6}{4} = -\dfrac{3}{2}$

$(0, -\tfrac{3}{2})$

(b) $r(x) = \dfrac{(x+4)(x-2)^2}{(x-3)(x+1)(x+5)^2}$



No holes

VA: $x = 3, \quad x = -1, \quad x = -5$

HA: $r(x) = \dfrac{x^3 + \cdots}{x^4 + \cdots}$

Deg Num = 3 < Deg Den = 4

$\underline{y = 0}$

x-int: $x = -4, 2$

$(-4, 0)$; $(2, 0)$

y-int: $r(0) = \dfrac{(4)(-2)^2}{(-3)(1)(5)^2}$

$= \dfrac{16}{-75}$

$\left(0, \dfrac{-16}{75}\right)$

	−5	−4	−1	2	3		
−	−	+	+	+	+	$(x+4)$
+	+	+	+	+	+	$(x-2)^2$
−	−	−	−	−	+	$(x-3)$
−	−	−	+	+	+	$(x+1)$
+	+	+	+	+	+	$(x+5)^2$
−	−	+	−	−	+	

Your instructor may or may not have covered the following:

9. Which of the rational functions in Problems 7 and 8 has a slant (oblique) asymptote? What is its slant (oblique) asymptote?

7b $\rightarrow$ Since degree numerator is exactly 1 more than degree of denominator, there is a slant asymptote.

$$\begin{array}{r}
\frac{1}{2}x + 3 \\
2x^2-8x-10 \;\big)\; x^3 + 2x^2 - 8x \\
-(x^3 - 4x^2 - 5x) \\
\hline
6x^2 - 3x + 0 \\
-(6x^2 - 24x - 30) \\
\hline
21x + 30
\end{array}$$

$$\frac{x^3+2x^2-8x}{2x^2-8x-10} = \underbrace{\frac{1}{2}x + 3} + \underbrace{\frac{21x+30}{2x^2-8x-10}}_{\rightarrow 0 \text{ as } x \rightarrow \pm\infty}$$

Slant Asymptote: $y = \frac{1}{2}x + 3$

10. Use synthetic division to find the quotient and remainder when $x^3 - 8x - 5$ is divided by $x + 3$.

$$\begin{array}{r|rrrr} -3 & 1 & 0 & -8 & -5 \\ & \downarrow & -3 & 9 & -3 \\ \hline & 1 & -3 & 1 & -8 \end{array}$$

$x - c$

$x - (-3)$

$\boxed{Q(x) = x^2 - 3x + 1}$ $\boxed{R(x) = -8}$

11. Factor the polynomial $P(x) = 2x^3 - 3x^2 - 17x + 30$ completely if it is known that $\frac{5}{2}$ is a zero of the polynomial.

$\frac{5}{2}$ is a zero $\Longrightarrow$ $(x - 5/2)$ is a factor.

$P(x)$ divide by $x - 5/2$

$$
\begin{array}{c|cccc}
5/2 & 2 & -3 & -17 & 30 \\
 & \downarrow & 5 & 5 & -30 \\
\hline
 & 2 & 2 & -12 & \boxed{0}
\end{array}
$$

$$
\begin{aligned}
P(x) &= (x - 5/2)(2x^2 + 2x - 12) \\
&= 2(x - 5/2)(x^2 + x - 6) \\
&= \boxed{2(x - 5/2)(x + 3)(x - 2)} \\
&= \boxed{(2x - 5)(x + 3)(x - 2)}
\end{aligned}
$$