10.1 Sequences

A sequence is an ordered list of numbers: \( a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots \)

Each of the numbers is called a term of the sequence.

Notation: A sequence \( \{a_1, a_2, a_3, \ldots\} \) can be denoted by \( \{a_n\} \) or \( \{a_n\}_{n=1}^{\infty} \). Unless otherwise stated, we assume \( n \) starts at 1, but this does not always have to be the case.

Examples: Find the first 4 terms of the following sequences.

\[ \left\{ \frac{n^2}{n+4} \right\}_{n=1}^{\infty} \]

- \( a_1 = \frac{1}{5} \)
- \( a_2 = \frac{4}{6} \)
- \( a_3 = \frac{9}{7} \)
- \( a_4 = \frac{16}{8} \)

\[ a_n = \left(\frac{1}{2}\right)^n \cdot n^3 \]

- \( a_1 = \left(-\frac{1}{2}\right)^1 \cdot 1^3 = -1 \)
- \( a_2 = \left(-\frac{1}{2}\right)^2 \cdot 2^3 = 8 \)
- \( a_3 = \left(-\frac{1}{2}\right)^3 \cdot 3^3 = -27 \)
- \( a_4 = \left(-\frac{1}{2}\right)^4 \cdot 4^3 = 64 \)
The above sequence is called an **alternating sequence** since the terms alternate signs.

Find a general formula \( a_n \) for the sequences below.

- \[
\begin{aligned}
&\{3, 4, 5, 6, \ldots \} \\
&\{4, 9, 16, 25, \ldots \}
\end{aligned}
\]

\[
\begin{align*}
\frac{3}{2} & \quad \frac{4}{3} & \quad \frac{5}{4} & \quad \frac{6}{5} \\
\frac{9}{4} & \quad \frac{16}{9} & \quad \frac{25}{16}
\end{align*}
\]

\[
\begin{aligned}
a_n &= \frac{n+2}{(n+1)^2} \\
\end{aligned}
\]

- \[
\begin{aligned}
&\{\frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \ldots \} \\
&\{\frac{4}{8}, \frac{8}{16}, \frac{16}{32}, \ldots \}
\end{aligned}
\]

\[
\begin{align*}
\frac{2}{3} & \quad \frac{3}{5} & \quad \frac{4}{7} & \quad \frac{5}{9} \\
\frac{4}{8} & \quad \frac{8}{16} & \quad \frac{16}{32}
\end{align*}
\]

\[
\begin{aligned}
a_n &= (-1)^n \cdot 2^n \\
\frac{2n+1}{2n+1} &= (-2)^n \\
\end{aligned}
\]

- \[
\begin{aligned}
&\{1, -2, 6, -24, 120, \ldots \} \\
&\{1!, -2!, 6!, -24!, 120!, \ldots \}
\end{aligned}
\]

\[
\begin{aligned}
a_n &= (-1)^{n+1} \cdot n! \\
\end{aligned}
\]
The limit of a sequence is defined to be \( \lim_{n \to \infty} a_n \).

If the limit exists to a finite number \( L \), i.e., \( \lim_{n \to \infty} a_n = L \), we say the sequence converges to \( L \). If the limit does not exist or is infinite, we say the sequence diverges.

Determine whether the following sequences converge or diverge. If the sequence converges, state the limit.

- \( a_n = 2 + \left( \frac{1}{3} \right)^n \)
  \[
  \lim_{n \to \infty} \left[ 2 + \left( \frac{1}{3} \right)^n \right] = 2 + 0 = \boxed{2 \to \text{Converges}}
  \]

- \( a_n = e^{\frac{n^2+1}{n+3}} \)
  \[
  \lim_{n \to \infty} e^{\frac{n^2+1}{n+3}} = \text{"} e^{\infty} \text{"} = \boxed{\infty \to \text{Diverges}}
  \]

- \( a_n = \arctan \left( \frac{-n^3 + 3}{\sqrt{2n + 1}} \right) \)
  \[
  \lim_{n \to \infty} \arctan \left( \frac{-n^3 + 3}{\sqrt{2n + 1}} \right) = \text{"} \arctan(-\infty) \text{"} = \boxed{-\frac{\pi}{2} \to \text{Converges}}
  \]
• $a_n = \frac{n}{\ln n}, \quad n \geq 2$

\[
\lim_{n \to \infty} \frac{n}{\ln n} = \lim_{n \to \infty} \frac{1}{1/n} = \lim_{n \to \infty} n = \infty \rightarrow \text{Diverges}
\]

• $a_n = \cos\left(\frac{3n + 7}{n^2 + n}\right)$

\[
\lim_{n \to \infty} \cos\left(\frac{3n + 7}{n^2 + n}\right) = \cos 0 = 1 \rightarrow \text{converges}
\]
\[ a_n = \frac{1}{2} \ln(3 + 7n^3) - \frac{1}{2} \ln(n^3 + 2n) = \frac{1}{2} \left[ \ln(3 + 7n^3) - \ln(n^3 + 2n) \right] \]

\[ \lim_{n \to \infty} \frac{1}{2} \ln \left( \frac{3 + 7n^3}{n^3 + 2n} \right)^{1/2} = \frac{1}{2} \ln 7 \quad \text{converges} \]

\[ a_n = (n - \sqrt{n^2 + 1}) \cdot \frac{n + \sqrt{n^2 + 1}}{n + \sqrt{n^2 + 1}} = \frac{n^2 - (n^2 + 1)}{n + \sqrt{n^2 + 1}} = -1 \]

\[ \lim_{n \to \infty} \frac{-1}{n + \sqrt{n^2 + 1}} = 0 \quad \text{converges} \]

\[ a_n = (-1)^n + 3 \]

\[ \begin{array}{c}
a_1 = -1 + 3 = 2 \\
a_2 = 4 \\
a_3 = 2 \\
a_4 = 4 \\
\vdots
\end{array} \]

\[ a_n = \sin(n\pi) \]

\[ \begin{array}{c}
a_1 = \sin(\pi) = 0 \\
a_2 = \sin(2\pi) = 0 \\
a_3 = \sin(3\pi) = 0
\end{array} \]

\[ a_n = \cos(n\pi) \]

\[ \begin{array}{c}
a_1 = \cos(\pi) = -1 \\
a_2 = \cos(2\pi) = 1 \\
a_3 = \cos(3\pi) = -1
\end{array} \]
For alternating sequences, use the following important fact: If \( \lim_{n \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

In other words, an alternating sequence converges if and only if the absolute value of the terms goes to 0 as \( n \) goes to \( \infty \).

- \( a_n = \frac{\cos(n\pi)}{n} \)
  - \( a_1 = \frac{\cos(1\pi)}{1} = -1 \)
  - \( a_2 = \frac{\cos(2\pi)}{2} = \frac{1}{2} \)
  - \( a_3 = \frac{\cos(3\pi)}{3} = -\frac{1}{3} \)
  - \( a_4 = \frac{1}{4} \)
  - \( a_5 = -\frac{1}{5} \)
  - \( a_n = (-3)^n = (-1)^n3^n \)
  - \( a_1 = -3 \)
  - \( a_2 = 9 \)
  - \( a_3 = -27 \)
  - \( a_5 = 81 \)

\[ \lim_{n \to \infty} |a_n| = 0 \]

So, \( \lim_{n \to \infty} a_n = 0 \) converges.

- \( a_n = (-1)^n \left( \frac{n^3 + 1}{n^4 + 3} \right) \)

\[ \lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{\frac{1}{4^n}}{\frac{3n^3}{n^4 + 3}} = 0 \]

So, \( \lim_{n \to \infty} a_n = 0 \) converges.

- \( a_n = (-1)^n \frac{n+1}{5n+2} \)

\[ \lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{n+1}{5n+2} = \frac{1}{5} \]

So, \( \lim_{n \to \infty} a_n \) DNE diverges.
A sequence \( \{a_n\} \) is said to be **increasing** if \( a_n < a_{n+1} \) for all \( n \geq 1 \).
A sequence \( \{a_n\} \) is said to be **decreasing** if \( a_n > a_{n+1} \) for all \( n \geq 1 \).

If a sequence is either an increasing or a decreasing sequence, we say it is **monotonic**.

A sequence \( \{a_n\} \) is **bounded** if there are numbers \( m \) and \( M \) such that \( m \leq a_n \leq M \) for all \( n \geq 1 \). We say that \( \{a_n\} \) is bounded above by \( M \) and bounded below by \( m \).

**Theorem:** Every bounded, monotonic sequence converges.

Determine whether the following sequences are increasing, decreasing, or not monotonic. Also state if the sequence is bounded or not.

\[ a_n = \frac{1}{2 + 5n} \]

\[ \lim_{n \to \infty} a_n = 0 \rightarrow \text{converges decreasing} \]

\[ Bounded: \quad 0 \leq a_n \leq \frac{1}{7} \]

\[ a_n = \frac{(-1)^n}{2 + 5n} \]

Since \( \lim_{n \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \rightarrow \text{converges} \]

\[ \text{Not monotonic} \]

\[ \text{Is bounded:} \quad -\frac{1}{12} \leq a_n \leq \frac{1}{12} \]
\[ a_n = n - \frac{1}{n} \]
\[ \lim_{n \to \infty} a_n = \infty \rightarrow \text{Diverges} \rightarrow \text{Not bounded} \]
\[ f(n) = n - \frac{1}{n} \quad f'(n) = 1 + \frac{1}{n^2} > 0 \rightarrow \text{increasing} \]

\[ a_n = e^{1/n} \]
\[ \lim_{n \to \infty} e^{\frac{1}{n}} = e^0 = 1 \rightarrow \text{converges} \]
\[ f(n) = e^{\frac{1}{n}} \rightarrow f'(n) = -\frac{1}{n^2}e^{\frac{1}{n}} < 0 \rightarrow \text{decreasing} \]
\[ \text{Bounded: } \quad 1 < a_n < e \]

\[ a_n = \sin\left(\frac{n\pi}{2}\right) \]
\[ \lim_{n \to \infty} a_n \quad \text{DNE} \rightarrow \text{oscillates} \rightarrow \text{diverges} \]
\[ a_1 = \sin\left(\frac{\pi}{2}\right) = 1 \quad \text{Not Monotonic} \]
\[ a_2 = \sin\left(\pi\right) = 0 \]
\[ a_3 = \sin\left(\frac{3\pi}{2}\right) = -1 \]
\[ a_4 = \sin\left(2\pi\right) = 0 \]

\[ a_n = \frac{2n}{n+1} \]
\[ \lim_{n \to \infty} \frac{2n}{n+1} = 2 \rightarrow \text{converges} \]
\[ f(n) = \frac{2n}{n+1} \rightarrow f'(n) = \frac{(n+1)2 - 2n(1)}{(n+1)^2} = \frac{2}{(n+1)^2} > 0 \rightarrow \text{increasing} \]
\[ \text{Bounded: } \quad 1 \leq a_n < 2 \]
A **recursive sequence** is one in which terms are defined by using previous terms in the sequence. In a recursive sequence, you have to be given at least the first term.

Find the next 4 terms of the recursively defined sequences. Do these sequences converge?

\[ a_1 = 1, \quad a_{n+1} = 4 - a_n \]

\[ a_1 = 1 \]
\[ a_2 = 4 - a_1 = 4 - 1 = 3 \]
\[ a_3 = 4 - a_2 = 4 - 3 = 1 \]
\[ a_4 = 4 - a_3 = 4 - 1 = 3 \]

Diverges by oscillation.

\[ a_1 = 3, \quad a_{n+1} = 2 + \frac{a_n}{3} \]

\[ a_2 = 2 + \frac{a_1}{3} = 2 + \frac{3}{3} = 3 \]
\[ a_3 = 2 + \frac{a_2}{3} = 2 + \frac{3}{3} = 3 \]
\[ a_4 = 3 \]

\[ a_n \text{ converges to } 3. \]

The sequence below is bounded and decreasing. Find the next two terms of the sequence and find the limit.

\[ a_1 = 4, \quad a_{n+1} = \frac{12}{8 - a_n} \]

\[ a_2 = \frac{12}{8 - a_1} = \frac{12}{8 - 4} = 3 \]
\[ a_3 = \frac{12}{8 - a_2} = \frac{12}{8 - 3} = \frac{12}{5} \]

\[ a_n \text{ must converge to some limit } L. \]

\[ a_{n+1} = \frac{12}{8 - a_n} \]
\[ L = \frac{12}{8 - L} \]
\[ L(8 - L) = 12 \]
\[ 8L - L^2 = 12 \]
\[ 0 = L^2 - 8L + 12 \]
\[ 0 = (L - 6)(L - 2) \]
\[ L = 6, 2 \]

Since \( a_n \) is decreasing and \( a_1 = 4 \), then \( L = 2 \).
10.2 Series

A series is the sum of an infinite sequence of numbers. So, given a sequence \( \{a_n\} \), the series \( \sum_{n=1}^{\infty} a_n \) is defined to be

\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots
\]

Can an infinite sequence of numbers have a finite sum? Sometimes yes and sometimes no! (This is similar to improper integrals where an infinite interval may or may not have a finite area.)

If \( \sum_{n=1}^{\infty} a_n = s \) where \( s \) is a finite number, then the series converges to \( s \) and we say \( s \) is the sum of the series. If the sum does not exist (or is infinite), the series diverges.

Given a series \( \sum_{n=1}^{\infty} a_n \), the \( n \)th partial sum, denoted \( s_n \), is defined to be the sum of the first \( n \) terms.

\[
s_1 = a_1
\]
\[
s_2 = a_1 + a_2
\]
\[
s_3 = a_1 + a_2 + a_3
\]

In general, \( s_n = a_1 + a_2 + a_3 + \ldots + a_n \)

For the following series, find the first 4 partial sums.

- \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \)
  \[
  s_1 = 1 \quad s_2 = 2 \quad s_3 = 3 \quad s_4 = 4
  \]

- \( \sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + \ldots \)
  \[
  s_1 = -1 \quad s_2 = 0 \quad s_3 = 1 \quad s_4 = 0
  \]

- \( \sum_{n=1}^{\infty} 3^n = 3 + 9 + 27 + 81 + 243 + \ldots \)
  \[
  s_1 = 3 \quad s_2 = 12 \quad s_3 = 39 \quad s_4 = 120
  \]

- \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \)
  \[
  s_1 = 1 \quad s_2 = \frac{3}{2} \quad s_3 = \frac{11}{6} \quad s_4 = \frac{15}{12}
  \]
Consider the sequence of partial sums, \( \{s_n\} = \{s_1, s_2, s_3, \ldots \} \). If this infinite sequence of partial sums converges to a number, that number is the sum of the series. In other words, a series converges if its sequence of partial sums converges, and

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n
\]

Example: Suppose for the series \( \sum_{n=1}^{\infty} a_n \), it is known that \( s_n = \frac{3n - 1}{2n + 6} \).

What is the sum of the first 5 terms?

\[
S_5 = \frac{14}{16} = \frac{7}{8}
\]

What is the sum of the first 10 terms?

\[
S_{10} = \frac{29}{26}
\]

Does the series converge or diverge? If it converges, what is the sum?

\[
\sum a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{3n-1}{2n+6} = \frac{3}{2} \quad \text{series converges to a sum of } \frac{3}{2}.
\]

What is \( a_1 \)?

\[
S_1 = a_1
\]

\[
a_1 = \frac{2}{8} = \frac{1}{4}
\]

What is \( a_8 \)?

\[
a_8 = S_8 - S_7 = \frac{23}{22} - \frac{20}{20} = \frac{1}{22}
\]

Find a formula for \( a_n \) for \( n > 1 \).

\[
a_n = S_n - S_{n-1} = \frac{3n-1}{2n+6} - \frac{3(n-1)-1}{2(n-1)+6} = \frac{3n-1}{2n+6} - \frac{3n-4}{2n+4}
\]

Example: Suppose for the series \( \sum_{n=1}^{\infty} a_n \), it is known that \( s_n = 2 + \left(\frac{3}{2}\right)^n \). Does the series converge or diverge? If it converges, what is the sum?

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left[2 + \left(\frac{3}{2}\right)^n\right] = \infty \quad \text{Series Diverges!}
\]
In general, finding a formula for $s_n$ is not easy. However, a type of series for which a formula for $s_n$ can often be determined are **telescoping series**. Telescoping series are series in which all but a finite number of terms of the series cancel out with other terms.

Examples: Find a formula for the $n$th partial sum for the following series. Does the series converge? If so, what is its sum?

$$\sum_{n=1}^{\infty} \left( \frac{2}{n+2} - \frac{2}{n+3} \right)$$

$$S_n = \left( \frac{2}{3} - \frac{2}{4} \right) + \left( \frac{2}{5} - \frac{2}{6} \right) + \ldots + \left( \frac{2}{n+1} - \frac{2}{n+2} \right) + \frac{2}{n+3}$$

$$S = \lim_{n \to \infty} \left( \frac{2}{3} - \frac{2}{n+3} \right) = \left[ \frac{2}{3} \right] \rightarrow \text{converges}$$
\[
\sum_{n=2}^{\infty} \ln \left( \frac{n+2}{n+1} \right) = \sum_{n=2}^{\infty} \left[ \ln(n+2) - \ln(n+1) \right]
\]

\[
S_n = \left( \ln 4 - \ln 3 \right) + \left( \ln 5 - \ln 4 \right) + \left( \ln 6 - \ln 5 \right) + \cdots + \left( \ln(n+1) - \ln n \right) + \left[ \ln(n+2) - \ln(n+1) \right]
\]

\[
S_n = -\ln 3 + \ln(n+2)
\]

\[
s = \lim_{n \to \infty} \left[ -\ln 3 + \ln(n+2) \right] \to \infty \quad \text{Series Diverges}
\]
\[
\sum_{n=3}^{\infty} \left( \cos \frac{1}{n-1} - \cos \frac{1}{n+1} \right)
\]

\[
S_n = \cos \frac{1}{2} - \cos \frac{1}{3} + \cos \frac{1}{3} - \cos \frac{1}{4} + \cos \frac{1}{4} - \cos \frac{1}{5} + \cos \frac{1}{5} - \cos \frac{1}{6} + \ldots
\]

\[
S_n = \cos \frac{1}{2} + \cos \frac{1}{3} - \cos \frac{1}{n} - \cos \frac{1}{n+1}
\]

\[
\lim_{n \to \infty} \left[ \cos \frac{1}{2} + \cos \frac{1}{3} - \cos \frac{1}{n} - \cos \frac{1}{n+1} \right] = \cos \frac{1}{2} + \cos \frac{1}{3} - \cos 0 - \cos 0 = \cos \frac{1}{2} + \cos \frac{1}{3} - 2
\]
If a series is not always written in a telescoping form, partial fractions may sometimes be used to write it in telescoping form.

\[
\sum_{n=4}^{\infty} \frac{-2}{n^2 - 2n} = \sum_{n=4}^{\infty} \frac{-2}{n(n-2)} \cdot \frac{n(n-2)}{(n-1)} = \left[ \sum_{n=4}^{\infty} \frac{-2}{n(n-2)} \right] \cdot \frac{A}{n} + \frac{B}{n-2}
\]

\[-2 = A(n-2) + Bn\]

\[n=2: \quad -2 = 2B \Rightarrow B = -1\]

\[n=0: \quad -2 = -2A \Rightarrow A = 1\]

\[
S_n = \left( \frac{1}{4} - \frac{1}{2} \right) + \left( \frac{1}{6} - \frac{1}{3} \right) + \left( \frac{1}{10} - \frac{1}{5} \right) + \ldots + \left( \frac{1}{n-1} - \frac{1}{n} \right)
\]

\[S = \lim_{n \to \infty} \left[ \frac{1}{2} - \frac{1}{3} + \frac{1}{x-1} + \frac{1}{n} \right] = -\frac{1}{2} - \frac{1}{3} = \left[ \frac{-5}{6} \right]
\]
If we can find a formula for $s_n$ for a series, we can determine if the series converges or diverges, but this is usually not easy. If you are not given a formula for $s_n$ or cannot determine one, how can you determine if a series converges or diverges? There are many tests we will learn. The first is the Test for Divergence.

**Test for Divergence:** If $\lim_{n \to \infty} a_n \neq 0$ or does not exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

If a series $\sum_{n=1}^{\infty} a_n$ converges, then it first must be true that $\lim_{n \to \infty} a_n = 0$.

**However, if** $\lim_{n \to \infty} a_n = 0$, **this DOES NOT NECESSARILY MEAN THE SERIES CONVERGES!!!**

The classic example of a series that does NOT converge even though its terms go to 0 is the harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$. However, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ DOES converge!

What can you conclude about the following series with regards to convergence/divergence using the Test for Divergence?

$$\sum_{n=1}^{\infty} \frac{n^2}{n + 7} \quad \lim_{n \to \infty} \frac{n^2}{n + 7} = \infty \rightarrow \text{Series Diverges by T.D.}$$

$$\sum_{n=1}^{\infty} \cos(n \pi) \quad \lim_{n \to \infty} \cos(n \pi) = \text{DNE} \rightarrow \text{Series Diverges by T.D.}$$

$$\sum_{n=1}^{\infty} \frac{n + 3}{n^2 + 5} \quad \lim_{n \to \infty} \frac{n + 3}{n^2 + 5} = 0 \rightarrow \text{T.D. is inconclusive.}$$

$$\sum_{n=1}^{\infty} \frac{n + 9}{3n - 7} \quad \lim_{n \to \infty} \frac{n + 9}{3n - 7} = \frac{1}{3} \neq 0 \rightarrow \text{Series Diverges by T.D.}$$

$$\sum_{n=1}^{\infty} \arctan(e^{-n}) \quad \lim_{n \to \infty} \arctan(e^{-n}) = 0 \rightarrow \text{T.D. is inconclusive}.$$
A geometric series is a series in which each term of the series is some ratio \( r \) times the previous term, where \( r \) is some real number. These types of series can be represented as \( \sum_{n=0}^{\infty} ar^n \).

It can be shown that for a geometric series of this form, \( s_n = \frac{a - r^n}{1 - r} \).

\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a - r^n}{1 - r} \text{ converges to a finite number when } |r| < 1, \text{ or equivalently, when } |r| < 1. 
\]

In this case, the sum of the geometric series is \( \frac{a}{1 - r} \), where \( a \) is the first term of the series.

Note that not all series start at \( n = 0 \). No matter what number the index starts with, \( a \) is still the first term of the series.

Examples: For each of the following series, determine if the series converges or diverges. If it converges, find the sum.

\[
5 - \frac{25}{6} + \frac{125}{36} - \frac{625}{216} + \cdots + \quad r = -\frac{5}{6} \quad [\text{Every term is } -\frac{5}{6} \text{ times previous term.}]
\]

Since \( |r| < 1 \), series converges.

\[
\text{Sum} = \frac{a}{1 - r} = \frac{5}{1 - (-\frac{5}{6})} = \frac{5}{\frac{11}{6}} = \frac{30}{11}
\]

\[
\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = 5 + 5\left(\frac{3}{4}\right) + 5\left(\frac{3}{4}\right)^2 + 5\left(\frac{3}{4}\right)^3 + \cdots
\]

\( r = \frac{3}{4} \rightarrow \) Since \( |r| < 1 \), series converges.

\[
a = 5
\]

\[
\text{Sum} = \frac{5}{1 - \frac{3}{4}} = \frac{5}{\frac{1}{4}} = 20
\]

\[
\sum_{n=1}^{\infty} \frac{4(-1)^n}{3^{2n}} = \sum_{n=1}^{\infty} \frac{4(-1)^n}{9^n} = \sum_{n=1}^{\infty} \left(\frac{-1}{9}\right)^n
\]

\( r = -\frac{1}{9} \rightarrow |r| < 1 \quad \checkmark \text{ Series converges.} \)

\[
a = 4(-\frac{1}{9}) = -\frac{4}{9}
\]

\[
\text{Sum} = \frac{-\frac{4}{9}}{1 - (-\frac{1}{9})} = \frac{-\frac{4}{9}}{\frac{10}{9}} = \frac{-\frac{4}{9}}{\frac{10}{9}} = \frac{-4}{10} = \frac{-2}{5}
\]

\[
3^{2n} = (3^2)^n = 9^n
\]
\[ \sum_{n=2}^{\infty} (-2)^{n+1} \frac{1-n}{5^n} = \sum_{n=2}^{\infty} (-2) \left( \frac{2}{5} \right)^n \]
\[ r = -\frac{2}{5} \]
\[ |r| < 1 \checkmark \]

Series converges.

\[ a = -2 \left( \frac{2}{5} \right)^2 = -2 \left( \frac{4}{25} \right) = -\frac{8}{25} \]

\[ \text{Sum} = \frac{-\frac{8}{25}}{1 - \left( -\frac{2}{5} \right)} = \frac{-\frac{8}{25}}{\frac{3}{5}} = -\frac{8}{15} \]

\[ \sum_{n=1}^{\infty} -\frac{5^{2n+1}}{6^n - 1} = \sum_{n=1}^{\infty} -\frac{5^{2n}}{6^n} \cdot \frac{5}{6} = \sum_{n=1}^{\infty} -\frac{25^n \cdot 5 \cdot 5}{6^n} = \sum_{n=1}^{\infty} -30 \left( \frac{5}{6} \right)^n \]
\[ r = \frac{25}{6} > 1 \rightarrow \text{Series Diverges} \]

\[ \sum_{n=0}^{\infty} \frac{-3 + e^n}{2(7^n+1)} = \sum_{n=0}^{\infty} \frac{-3}{2(7^n+1)} + \sum_{n=0}^{\infty} \frac{e^n}{2(7^n+1)} \]
\[ = \sum_{n=0}^{\infty} \frac{-3}{2 \cdot 7^n} + \sum_{n=0}^{\infty} \frac{e^n}{2 \cdot 7^n} \]
\[ = \sum_{n=0}^{\infty} \frac{-3}{14} \left( \frac{1}{7} \right)^n + \sum_{n=0}^{\infty} \frac{1}{14} \left( \frac{e}{7} \right)^n \]
\[ r = \frac{1}{7} \checkmark \quad |r| < 1 \]
\[ r = \frac{e}{7} \quad |r| < 1 \checkmark \]

\[ = \frac{-3/4}{1 - 1/7} + \frac{1/4}{1 - e/7} \]
For what values of \( x \) do the following series converge? For these values of \( x \), find the sum of the series.

\[
\sum_{n=1}^{\infty} \frac{x^n}{(-8)^{n+1}} = \sum_{n=1}^{\infty} \frac{x^n}{(-8)(-8)^n} = \sum_{n=1}^{\infty} -\frac{1}{8} \left( \frac{\alpha}{8} \right)^n
\]

\[
r = -\frac{\alpha}{8} \quad \rightarrow \text{For convergence, need} \quad \left| r \right| < 1
\]

\[
|\frac{-\alpha}{8}| < 1
\]

\[
\left| \frac{\alpha}{8} \right| < 1
\]

\[
|\alpha| < 8
\]

\[
-8 < \alpha < 8
\]

\text{Interval: } (-8, 8)

For these values, the sum is

\[
\sum_{n=0}^{\infty} \frac{2^{n+1}(x+3)^n}{3(4^n)} = \sum_{n=0}^{\infty} \frac{2 \cdot 2^n (x+3)^n}{3 \cdot 4^n} = \sum_{n=0}^{\infty} \frac{2}{3} \left( \frac{2(x+3)}{4} \right)^n
\]

\[
r = \frac{2(x+3)}{4} \quad \rightarrow \text{Need} \quad \left| r \right| < 1
\]

\[
\left| \frac{2(x+3)}{4} \right| < 1
\]

\[
\left| 2(x+3) \right| < 4
\]

\[
\left| x+3 \right| < 2
\]

\[
-2 < x + 3 < 2
\]

\[-5 < x < -1
\]

\text{Interval: } (-5, -1)
So far, the only series we can actually find a sum for are:

- Series for which we are given $s_n$ or can easily find it.
- Convergent telescoping series.
- Convergent geometric series.