8.1 Integration by Parts

Recall the Product Rule. If \( u \) and \( v \) are functions of \( x \), using differential notation, we have:

\[
(uv)' = u'v + uv' \\
(uv)' = udv + v du
\]

Integrating both sides with respect to \( x \), we would get:

\[
uv = \int [udv + v du] = \int udv + \int v du
\]

Rearranging this, gives us the following:

\[
\int udv = uv - \int v du
\]

Integration by parts is most often used when the integrand is a product of two different types of functions, like polynomial and exponential, or polynomial and trig, etc.

Integrals Involving Polynomials and Exponential Functions:

- \( \int_0^1 xe^{3x} \, dx \)
- \( \int (x^2 + 5)e^{-2x} \, dx \)
In general, before integrating by parts, the argument or “inside” of the exponential function must be linear, otherwise you won’t be able to integrate it to find $v$.

- $\int x^7 e^{-x^4} \, dx$

Integrals Involving Polynomials and Trigonometric Functions:

- $\int 3x \sin 2x \, dx$

- $\int x^5 \cos(x^2) \, dx$
Integrals Involving Polynomials and Logarithmic Functions:

- \( \int \ln x \, dx \)
- \( \int x^7 \ln x \, dx \)
- \( \int \frac{\ln x}{x^2} \, dx \)

A useful acronym to help remember what to let \( u \) be is **L I P E T**.
Integrals Involving Inverse Trig Functions:

- $\int \arcsin(x) \, dx$

Loops:

- $\int e^x \cos 2x \, dx$

Another type of integral that would require a loop would be $\int \cos(3x) \sin(2x) \, dx$
8.2 Trig Integrals

VERY IMPORTANT identities for this section:

\[ \sin^2 x + \cos^2 x = 1 \] which means that \[ \sin^2 x = 1 - \cos^2 x \] and \[ \cos^2 x = 1 - \sin^2 x \]

\[ \tan^2 x + 1 = \sec^2 x \] which means that \[ \tan^2 x = \sec^2 x - 1 \]

\[ \cot^2 x + 1 = \csc^2 x \] which means that \[ \cot^2 x = \csc^2 x - 1 \]

Goal of Trig Integrals: Choose \( u \) so that when you factor out its derivative for \( du \), whatever is left can be rewritten in terms of \( u \) using trig identities. This is achieved by making sure that whatever “non-\( u \)” stuff is left involves even powers.

Integrals Involving Powers of Sines and Cosines: If there is an odd power of sine or cosine in the original integral, let \( u \) be the other trig function. (Note: If both powers are odd, \( u \) can be either trig function.)

- \[ \int \sin^3 x \cos^6 x \, dx \]

- \[ \int \sin^6 x \cos^5 x \, dx \]

- \[ \int \cos^3 x \, dx \]
• \( \int \sin(5x) \tan^2(5x) \, dx \)

WHEN ALL THE POWERS OF SINE AND COSINE IN THE ORIGINAL INTEGRAL ARE EVEN, YOU MUST USE THE FOLLOWING IDENTITIES!

\[
\cos^2 x = \frac{1}{2} (1 + \cos 2x) \\
\sin^2 x = \frac{1}{2} (1 - \cos 2x)
\]

• \( \int \sin^2 x \, dx \)

• \( \int \sin^2 x \cos^2 x \, dx \)

To integrate \( \int \cos^4 x \, dx \) or \( \int \sin^4 x \, dx \), treat it like above as \( \int \cos^2 x \cos^2 x \, dx \) or \( \int \sin^2 x \sin^2 x \, dx \) respectively. Same idea with even higher powers.
Integrals Involving Powers of Secant and Tangent: If the power of secant is even, let $u$ be tangent. If the power of tangent is odd, let $u$ be secant. If both are true, either will work. If neither is true, it’s more difficult.

- $\int \tan^8 x \sec^4 x \, dx$

- $\int x \tan^5(x^2) \sec^3(x^2) \, dx$

- $\int \frac{\sec^4(\ln x)}{x} \, dx$
\[ \int \tan^2 x \, dx \]

\[ \int \cot^3 x \csc^3 x \, dx \]

\[ \int \sec x \, dx \]