8.9 Improper Integrals

An integral is said to be “improper” when either:

• The interval over which you are integrating is infinite.
• The function has a discontinuity somewhere in the interval over which you are integrating.

**Type 1: Infinite Intervals**

On an interval \([a, \infty]\):

\[
\int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx
\]

On an interval \([-\infty, b]\):

\[
\int_{-\infty}^b f(x) \, dx = \lim_{t \to -\infty} \int_t^b f(x) \, dx
\]

The integral is said to **converge** if the limit exists and **diverge** if the limit does not exist.

On the interval \([-\infty, \infty]\), we split up the integral into two separate improper integrals.

\[
\int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^a f(x) \, dx + \int_a^\infty f(x) \, dx
\]

The integral converges as long as BOTH integrals on the right converge.

If EITHER integral on the right diverges, then the original integral diverges.

Determine whether the following integrals converge or diverge. If the integral converges, find its value.

• \(\int_1^\infty \frac{1}{x^2} \, dx\)

• \(\int_1^{\infty} \frac{1}{x} \, dx\)

Fact of Mucho Importance: \(\int_1^\infty \frac{1}{x^p} \, dx\) converges ONLY when \(p > 1\). If \(p \leq 1\), it diverges.

The lower limit of integration does not have to be 1. This is actually true if the lower limit of integration is any positive number greater than 0.
\begin{itemize}
    \item \( \int_1^\infty \frac{\ln x}{x} \, dx \)
    \item \( \int_0^\infty x^2 e^{-x^3} \, dx \)
    \item \( \int_{-\infty}^0 xe^x \, dx \)
\end{itemize}
\[ \int_{5}^{\infty} \frac{1}{x^2 - 7x + 12} \, dx \]

\[ \int_{-\infty}^{\infty} \frac{1}{x^2 + 9} \, dx \]
Type 2: Discontinuous Functions on an Interval \([a, b]\)

If \(f(x)\) is continuous everywhere on the interval except at \(x = b\), then
\[
\int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx
\]

If \(f(x)\) is continuous everywhere on the interval except at \(x = a\), then
\[
\int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx
\]

The integral is said to **converge** if the limit exists and **diverge** if the limit does not exist.

If \(f(x)\) has a discontinuity at \(x = c\) somewhere in the interval \((a, b)\), then we split up the integral into two improper integrals.
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]

The integral converges as long as BOTH integrals on the right converge. If EITHER integral on the right diverges, then the original integral diverges.

- \(\int_0^2 \frac{1}{(x-2)^5} \, dx\)

- \(\int_{-3}^5 \frac{1}{\sqrt{x+3}} \, dx\)
\[
\int_{-4}^{6} \frac{7}{(x+2)^2} \, dx \\
\int_{0}^{2} x \ln x \, dx
\]

Comparison Test for Integrals

Suppose that \( f \) and \( g \) are positive continuous functions for all \( x \geq a \)

- If \( f(x) \leq g(x) \) and if \( \int_{a}^{\infty} g(x) \, dx \) converges, then \( \int_{a}^{\infty} f(x) \, dx \) also converges. If the bigger converges, so does the smaller.

- If \( f(x) \geq g(x) \) and if \( \int_{a}^{\infty} g(x) \, dx \) diverges, then \( \int_{a}^{\infty} f(x) \, dx \) also diverges. If the smaller diverges, so does the larger.
Determine whether the following integrals converge or diverge by comparison with an appropriate integral (if possible).

• \( \int_1^\infty \frac{6}{x^3 + 1} \, dx \)

• \( \int_2^\infty \frac{x^2}{\sqrt{x^3 - 1}} \, dx \)

• \( \int_3^\infty \frac{5 + e^{-x}}{x} \, dx \)

• \( \int_2^\infty \frac{\cos^2 x}{x \sqrt{x}} \, dx \)
\[ \int_2^\infty \frac{\sin x + 4}{x} \, dx \]

\[ \int_1^\infty \frac{1}{x + e^{2x}} \, dx \]