9.3 Arc Length

Any of these formulas can be used to find arc length, depending on how the curve is defined. Notice each has a different variable of integration.

If \( y = f(x) \), then the length of the curve from \( x = a \) to \( x = b \) is:

\[
L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

If \( x = g(y) \), then the length of the curve from \( y = c \) to \( y = d \) is:

\[
L = \int_c^d \sqrt{1 + [g'(y)]^2} \, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy
\]

If the curve is defined parametrically \( x = f(t) \), \( y = g(t) \) then the length of the curve from \( t = a \) to \( t = b \) is:

\[
L = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2} \, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

Make sure your limits of integration match the variable of integration!!

Examples:

- Find the length of the curve \( y = 1 + 6x^{3/2} \) from the point \((0, 1)\) to the point \((1, 7)\).
• Find the length of the curve $y = \ln(\cos x)$, $0 \leq x \leq \frac{\pi}{4}$.

• Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$, $1 \leq y \leq 2$. 
• Find the length of the curve $x = e^{3t} + e^{-3t}, y = 10 - 6t, 0 \leq t \leq 1$.

• Find the length of the curve $x = t^2 + 4, y = t^3 + 1$, from the point $(4, 1)$ to the point $(8, 9)$.

• Set up both a $dx$ and a $dy$ integral to find the length of the curve $y = \arctan x, 0 \leq x \leq 1$. 
9.4 Surface Area of Revolution

In general, the surface area of a solid of revolution is

\[ S = \int 2\pi r \, ds \]

where \( ds \) is arc length.

If revolving about the \( x \)-axis, the radius is \( y \), so:

\[ S = \int 2\pi y \, ds \]

If revolving about the \( y \)-axis, the radius is \( x \), so:

\[ S = \int 2\pi x \, ds \]

These integrals can be done with respect to \( x \), \( y \), or \( t \) depending on how the function is defined by using the appropriate arc length expression from the previous section.

**Be sure the WHOLE integral is written in terms of ONE variable and that your limits of integration match this variable before integrating!!**

- Find the surface area of the solid obtained by rotating the curve \( y = x^3 \), \( 0 \leq y \leq 8 \) about the \( x \)-axis.
Examples:

• Find the surface area of the solid obtained by rotating the curve \( x = \frac{1}{2}(y^2 + 2)^{3/2}, \ 0 \leq y \leq 1 \) about the \( x \)-axis.

• Find the surface area of the solid obtained by rotating the curve \( y = x^2 - \frac{1}{8}\ln x, \ 1 \leq x \leq 2 \) about the \( y \)-axis.
• Find the surface area of the solid obtained by rotating the curve \( x = 2t^2, \ y = \frac{2}{3}t^3 - 2t, \ 0 \leq t \leq 1 \) about the \( y \)-axis.

• Find the surface area of the solid obtained by rotating the curve \( x = \sqrt{1 + 6y}, \ 1 \leq y \leq 5 \) about the \( y \)-axis.