Form-factor expansion: some forgotten orbits

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σ^v is a unitary matrix

Form-factor:

$$K(n/B) = \frac{n^2}{B} \sum_{p,q \in P_n} A_p A_q^* \delta_{p,\pi(q)}$$

Where

p, q - periodic orbits of length n

$p = [v_1, v_2, \dots v_n]$

$A_p = \sigma^{v_2}_{v_1 v_3} \sigma^{v_3}_{v_2 v_4} \dots \sigma^{v_1}_{v_n v_2}$

q visits the same edges as p

Want to show:

BGS: $K(n/B) \to K_{RMT}(\tau)$

$B \to \infty$ while $\frac{n}{B} \to \tau$

TR symmetry $\Rightarrow$ $K_{RMT}(\tau) = K_{GOE}(\tau)$

$= 2\tau - \tau \ln(1+2\tau) \quad \tau<1$

$= 2\tau - 2\tau^2 + 2\tau^3 - \frac{4}{3}\tau^4 + \dots$

History of periodic orbit expansions

	Billiards etc	Graphs
diag appr.	Berry '85	Kottos-Smilansky '99 Tanner '01
$-2\tau^2$	Sieber-Richter '00 Sieber '01 (uniform) Spehner '03 Turek-Richter '03 (general)	B.-Schanz-Whitney '02 (general)
$2\tau^3$	Heusler-Müller- -Braun-Haake '04	B.-Schanz-Whitney '03 (unit) B.'04 (gen)
full	Müller-Heusler- -Braun-Haake -Altland '04-05	B.'06 (unif)
Related Work		Bolte-Harrison (spin) Nagao-Saito (weak mag.f.) Gnutzmann-Altland (SuSy) Winn, Keating, Marklof, B. ("intermediate") star graphs

Fundamental Problem 1:

Series is "absolutely divergent":

$$\frac{n^2}{B} \sum_{p,q} \left| A_p A_q^* \delta_{p,\pi(q)} \right| \to \infty \qquad \text{as } \underset{n=\tau B}{B \to \infty}$$

$\therefore$ One has to carefully ensure no "significant" number / type of orbits is ignored

<u>Basic idea</u>: classify all "transformations" producing q from p

(chop p into pieces and re-arrange)

Examples: * $p = 1\ 2$ $\quad q = 1\ \bar{2}$  

* $p = 1\ 2\ 3$ $\quad q = 1\ 3\ 2$

* $p = 1\ 2\ 3\ 4$ $\quad q = 1\ \bar{4}\ 3\ \bar{2}$  

"diagrams"

Fundamental Problem 2:

Not all pairs (p,q) can be uniquely attributed to a "diagram"

Examples: * $p = 1\ 2\ 3\ 4\ 5\ 3\ 6$ $\quad q = 1\ 2\ 3\ 5\ 4\ 3\ 6$

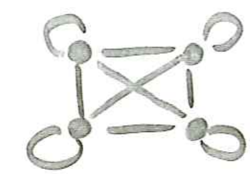
"local trouble" $\quad$ or

* $p = A\ A\ B\ B$ $\quad q = A\ B\ A\ B$

"global trouble" $\quad$ or

rare?

Test case: uniform complete graph



N vertices, Fourier σ^v

$$|\sigma^v_{ij}|^2 = \frac{1}{N}, \quad \sigma^v_{ij} = \sigma^v_{ji}$$

$B = N^2$ - number of directed bonds

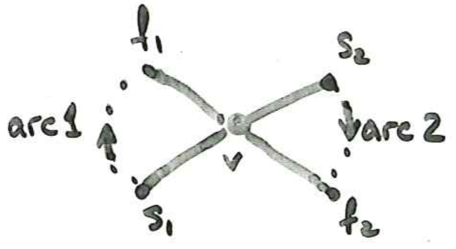
Classical analogue: Markov chain on bonds

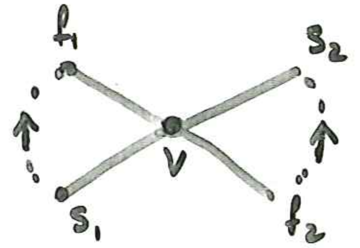
$$P_{(ij)(jk)} = |\sigma^j_{ik}|^2$$

Special about this graph: ultra-fast equilibration

$$P^n_{b_1 \to b_2} = \frac{1}{N^2} = \frac{1}{B} \quad \forall b_1, b_2 \quad n \geq 2$$

τ^2 Contribution:





Sum over:
* v
* s_1, f_1, s_2, f_2
* arc 1, arc 2

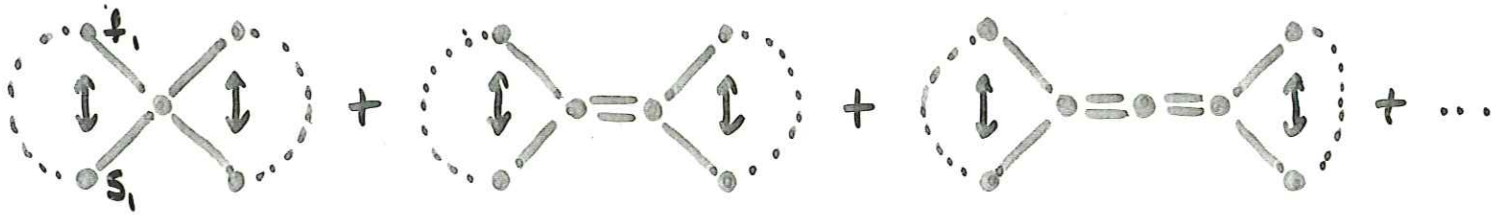
$$A_p = A_1 \sigma^v_{f_1 s_2} A_2 \sigma^v_{f_2 s_1} \qquad A_q = A_1 \underline{\sigma_{f_1 f_2}} A_{\bar{2}} \underline{\sigma_{s_2 s_1}}$$

$$A_{\bar{2}} = A_2$$

$$A_p A_q^* = |A_1|^2 |A_2|^2 \sigma_{f_1 s_2} \sigma_{f_2 s_1} \sigma^*_{f_1 f_2} \sigma^*_{s_2 s_1}$$

unitarity $\Rightarrow \sum_{f_1} \sigma_{f_1 s_2} \sigma^*_{f_1 f_2} = \delta_{s_2 f_2}$

Subdivide the orbits like this:



length (arc 1, 2) $\geq$ 1 bond $\Rightarrow$ $P_{(v s_1) \to (f_1 v)} = \frac{1}{B}$

independent of f_1

$\hookrightarrow$ sum over f_1 $= \delta_{s_2 t_2} \to 0$

$\Rightarrow$ $A_p A_q^* = |A_p|^2 = \frac{1}{N^n}$

Evaluating everything...

$$\frac{n^2}{B}\left[-(n-5)\frac{1}{N^2}\frac{N-1}{N} + (n-7)\frac{1}{N^2}\left(\frac{N-1}{N}\right)^2 + (n-9)\frac{1}{N^3}\left(\frac{N-1}{N}\right)^2 + \dots\right]$$

$$= \frac{n^2}{B}\left(\frac{N-1}{N}\right)^2\left[-(n-5)\frac{1}{N^2}\frac{1}{1-1/N} + (n-7)\frac{1}{N^2} + (n-9)\frac{1}{N^3} \dots\right]$$

$$= \frac{n^2}{B}\left(\frac{N-1}{N}\right)^2\left[\left((n-7)-(n-5)\right)\frac{1}{N^2} + \left((n-9)-(n-5)\right)\frac{1}{N^3} + \dots\right]$$

$$= -2\frac{n^2}{B^2} + O\left(\frac{1}{N}\right) \approx 0$$

Orbits of the type

(same for )

Naive estimate:

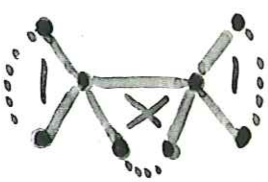
$$\frac{n^2}{B} \quad N^{n-2} \quad \frac{1}{N^n} \binom{n'}{2} \sim \frac{n^4}{N^4} = \tau^2 n^2 \nearrow \infty$$

N^{n-2}: # config.

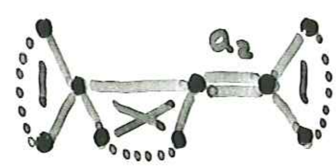
$\frac{1}{N^n}$: $|A_p A_q^*|$

$\binom{n'}{2}$: lengths of arcs $n' = n_1 + n_2 + n_3$

Proper evaluation:



$+ \sum_{a_1 \geq 1}$ 

$+ \sum_{a_2 \geq 1}$ 

$+ \sum_{a_1, a_2 \geq 1}$ 

( is counted as part of )

$$\frac{n^2}{N^2}\left[\binom{n'}{2}\frac{1}{N^4}\left(\frac{N-1}{N}\right)^2 - \sum_{a_1}\binom{n'-2a_1}{2}\frac{1}{N^{3+a_1}}\left(\frac{N-1}{N}\right)^3 - \sum_{a_2}\binom{n'-2a_2}{2}\frac{1}{N^{3+a_2}}\left(\frac{N-1}{N}\right)^3 + \sum_{a_1,a_2}\binom{n'-2a_1-2a_2}{2}\frac{1}{N^{2+a_1+a_2}}\left(\frac{N-1}{N}\right)^4\right]$$

$$= 4\tau^3 \cdot O\left(\frac{1}{\ }\right) \to 0$$

τ^3 contributor (one of):

Subdivide:

 $+\sum_{a_1 \geq 1}$ 

$+\sum_{a_2 \geq 1}$ 

$+\sum_{a_1, a_2 \geq 1}$ 

$=$ 

$$\frac{n^2}{B}\Bigg[\binom{n'}{3}\frac{1}{N^4}\left(\frac{N-1}{N}\right)^2$$

$-\sum_{a_1}$ 

$$-\sum_{a_1 \geq 1}\binom{n'-2a_1}{3}\frac{1}{N^{3+a_1}}\left(\frac{N-1}{N}\right)^3$$

$-\sum_{a_2}$ 

$$-\sum_{a_2 \geq 1}\binom{n'-2a_2}{3}\frac{1}{N^{3+a_2}}\left(\frac{N-1}{N}\right)^3$$

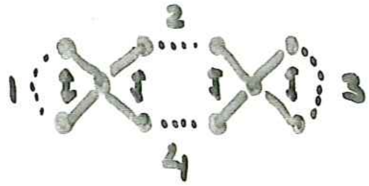
$+\sum_{a_1, a_2}$ 

$$+\sum_{a_1, a_2 \geq 1}\binom{n'-2a_1-2a_2}{3}\frac{1}{N^{2+a_1+a_2}}\left(\frac{N-1}{N}\right)^4\Bigg]$$

$$= 4\tau^3 + O\left(\frac{1}{N}\right)$$

$$= O\left(\frac{1}{N^2}\right)$$

But! (H. Schanz)

In 

we are implicitly assuming that arc 2 $\geq$ 1 bond
arc 4 $\geq$ 1 bond

<u>This</u> <u>assumption</u> <u>is</u> <u>"ok"</u> <u>because</u>:

* we get the right result
* everyone is doing it
* we are looking for universality and short arcs aren't universal

<u>The</u> <u>orbits</u> <u>that</u> <u>violate</u> <u>the</u> <u>assumption</u> <u>are</u> <u>interesting</u> <u>because</u>:

* they can - potentially - make a significant or even divergent contribution
* they are tricky and we are curious
* they can shed light on <u>deviations</u> from universality (or conditions for it)

Some ideas for non-uniform graphs:

$$\sum_{\text{arc}_j : b_s \to b_f} |A_{\text{arc}_j}|^2 = P^{n_j}_{b_s \to b_f} \to \frac{1}{B} \qquad n_j \to \infty$$

(mixing)

$$\left| \mathrm{Contr}_{\cancel{\text{unif}}}(n_1, n_2 ..) - \mathrm{Contr}_{\text{unif}}(n_1, n_2 ..) \right|$$

$$\leq B^r \underbrace{\max_{b_s, b_f} \left| P^{n_{min}}_{b_s \to b_f} - \frac{1}{B} \right|}_{\substack{\downarrow \\ 0}} \qquad n_{min} \to \infty$$

controlled by the spectrum of M. C.

$$\sum_{n_1, n_2 \ldots \geq 1} \mathrm{Contr}_{\cancel{\text{unif}}}(n_1, n_2 \ldots)$$

$$= \sum_{n_1, n_2 \ldots \geq n_{mix}} \mathrm{Contr}_{\cancel{\text{unif}}}(n_1, n_2 \ldots) + O(n_{mix}/n)$$

How does Gnutzmann-Altland gap condition arise?