$$\rho(\lambda) = \begin{cases} \frac{1}{\pi}\sqrt{\frac{2}{N}}\sqrt{1-\frac{\lambda^2}{2N}} & |\lambda| \leq (2N)^{1/2} \\ 0 & \end{cases}$$

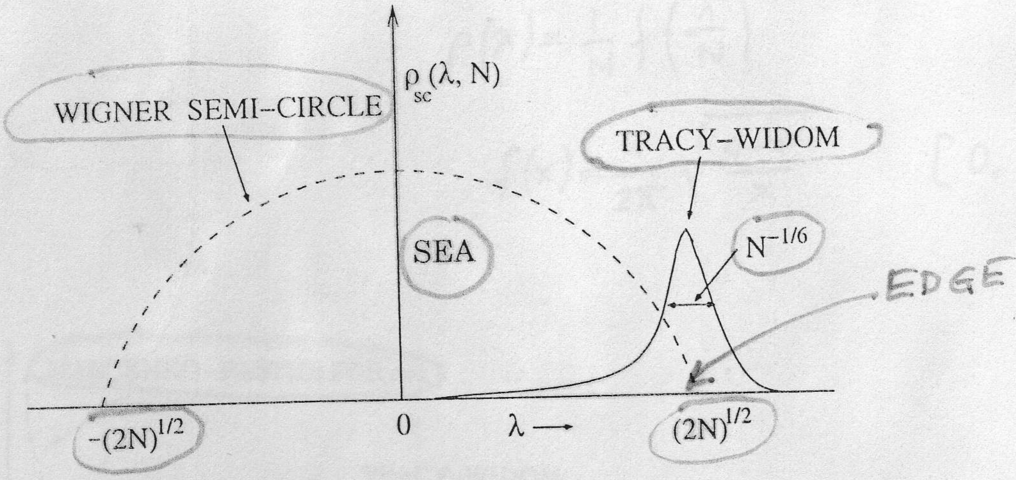


FIG. 1: The dashed line shows the semi-circular form of the average density of states. The largest eigenvalue is centered around its mean $\sqrt{2N}$ and fluctuates over a scale of width $N^{-1/6}$. The probability of fluctuations on this scale is described by the Tracy-Widom distribution (shown schematically).

scaling variable

$$\xi = \sqrt{2}\, N^{1/6} \left[\lambda_{max} - \sqrt{2N} \right]$$

$$\text{Prob}\left[\xi \leq x\right] = F_\beta(x) \leftarrow \text{has a limit } N \to \infty$$

(Tracy-Widom)

from Dean, Majumdar PRL 97 (2006) 160201

$$\rho(\lambda) = \frac{1}{N} f\left(\frac{\lambda}{N}\right)$$

$$f(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}} \qquad [0, 4]$$

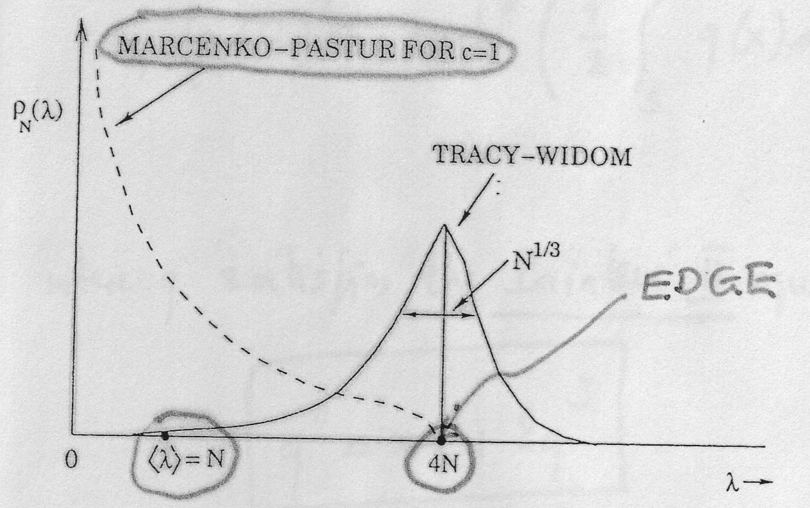


Figure 1. The dashed line schematically shows the Marčenko–Pastur form of the average density of states for $c = 1$. The average eigenvalue for $c = 1$ is $\langle\lambda\rangle = N$. For $c = 1$, the largest eigenvalue is centred around its mean $\langle\lambda_{\max}\rangle = 4N$ and fluctuates over a scale of width $N^{1/3}$. The probability of fluctuations on this scale is described by the Tracy–Widom distribution (shown schematically).

scaling variable

$$\xi = c^{-1/6} x_+^{-2/3} N^{-1/3} (\lambda_{max} - x_+ N)$$

$$x_+ = \left(\frac{1}{\sqrt{c}} + 1\right)^2, \quad c = \frac{N}{M}$$

$\text{Prob}(\xi \le x) = F_\beta(x)$ has a large N limit (Tracy–Widom)

Johansson
Johnston

From P. Vivo, S. Majumdar, O. Bohigas cond-mat/0701371
J. Phys. A 40 (2007) 4371

$\boxed{\text{F: distribution function of largest eigenvalue}}$ Tracy–Widom

(Dyson index)

1 $$F_1(s)^2 = \exp\left(-\int_s^\infty q(x)\,dx\right) \cdot F_2(s)$$

2 $$F_2(s) = \exp\left(-\int_s^\infty (x-s)\, q(x)^2\, dx\right)$$

4 $$F_4(s/\sqrt{2})^2 = \cosh^2\left(\frac{1}{2}\int_s^\infty q(x)\,dx\right) \cdot F_2(s)$$

where q satisfies the <u>Painlevé II</u> equation

$$\boxed{q'' = xq + 2q^3}$$

with boundary condition

Airy function

$$q(x) \sim -Ai(x) \quad \text{as } x \to \infty$$

$$f_\beta(s) = \frac{dF_\beta(s)}{ds}$$

probability density function

$$F_2(s) \to 1 - O\left(\exp\left[-4x^{3/2}/3\right]\right) \qquad x \to \infty$$

$$\to \exp\left[-|x|^3/12\right] \qquad x \to -\infty$$

C. A. Tracy,
H. Widom,
Comm. Math. Phys.
159 (1994) 151
177 (1996) 727

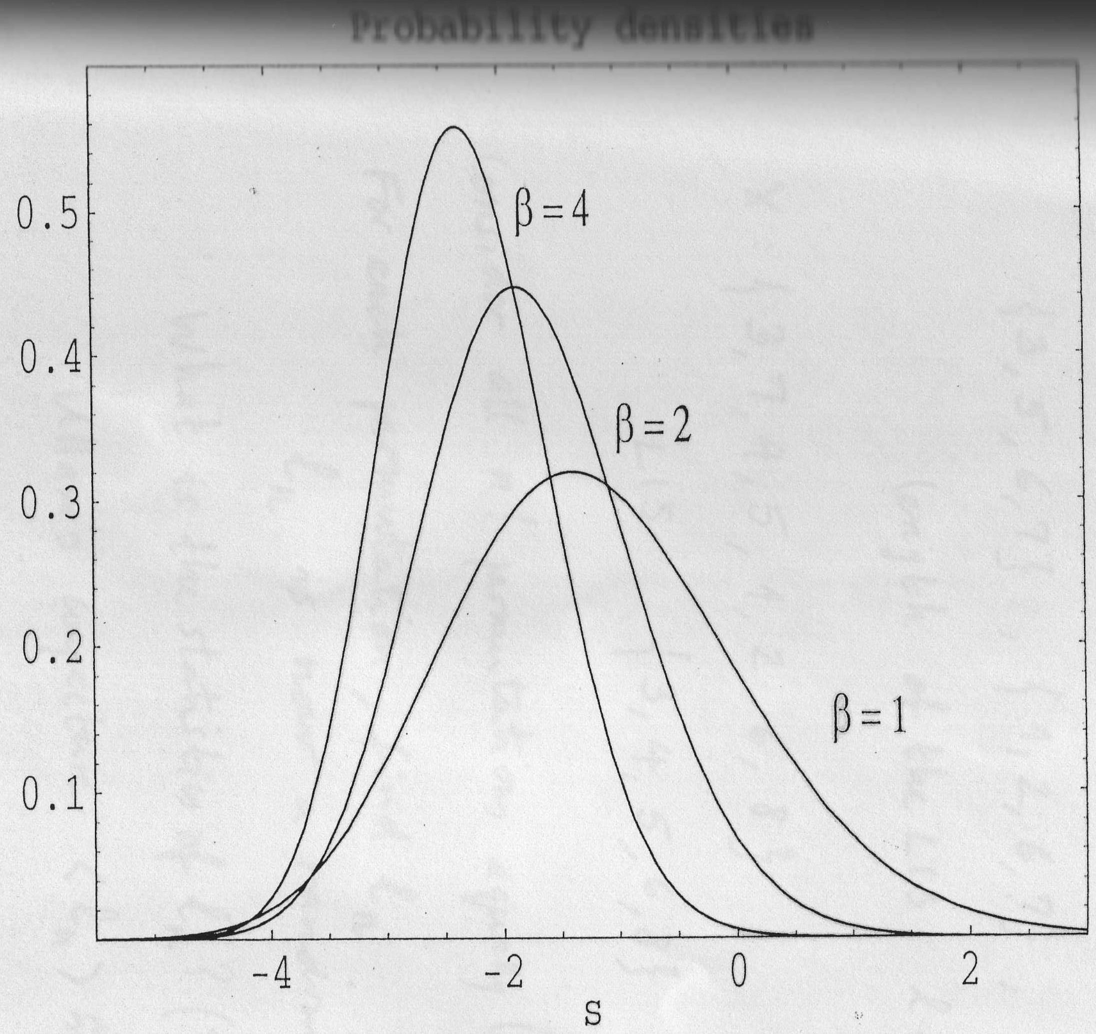


Figure 1: Densities for the scaled largest eigenvalues, $f_\beta(s)$.

LIS Longest increasing subsequence.

Sequence of n distinct integers

$X : \{8, 3, 5, 1, 2, 6, 4, 7\}$ $\quad n = 8$

subsequence : an ordered sublist of X : $\{3, 1, 2, 6\}$

longuest increasing subsequence

$\{3, 5, 6, 7\}$, $\{1, 2, 6, 7\}$, $\{1, 2, 4, 7\}$

length of the LIS $\quad \ell = 4$

$X : \{3, 7, 4, 5, 1, 2, 6, 8\}$

LIS $\quad \{3, 4, 5, 6, 8\} \quad \ell = 5$

Consider all $n!$ permutations equally likely (uniform measure)

For each permutation, find ℓ_n

ℓ_n is now a random variable

What is the statistics of ℓ_n? (Ullam's problem, 1961)

Ullam's conjecture $\langle \ell_n \rangle \approx a\sqrt{n}$

$a \approx 2$ ('68)

$a = 2$ (Vershik, 1977)

Longest increasing subsequence in a random permutation of $\{1, \dots, n\}$

L_n

For large n

$$\langle L_n \rangle = 2\sqrt{n}$$

Distribution of

$$(2\sqrt{n} - L_n)/n^{1/6}$$

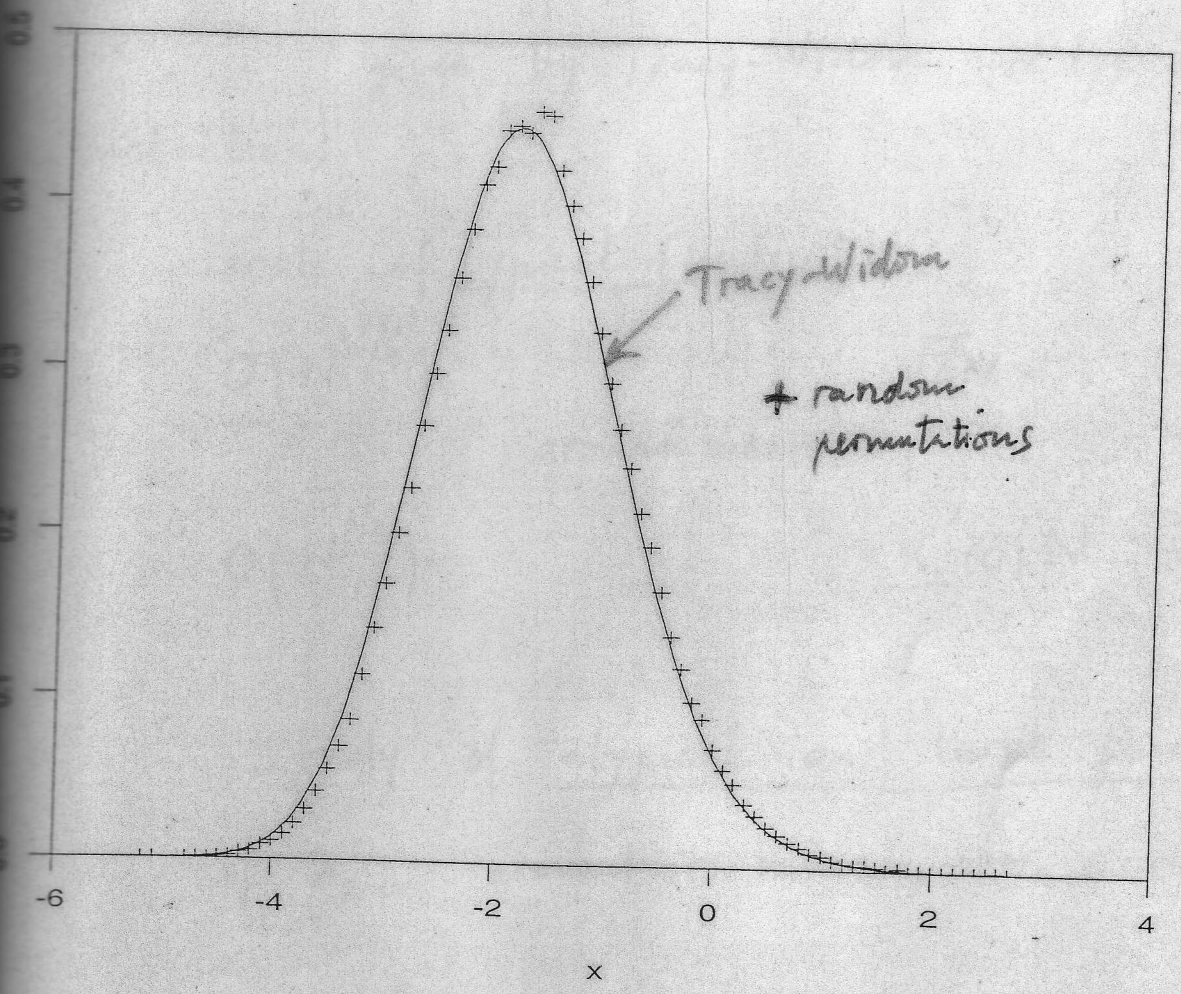
is Tracy-Widom !! $(\beta = 2)$

RANDOM COMBINATORICS

L_n : length of the longest increasing subsequence of a random permutation of $\{1, \dots, n\}$

$\langle L_n \rangle \sim 2\sqrt{n}$ for n large

detailed statistics on increasing subsequences



$\beta = 2$

Figure 1: Asymptotic density function for for $n = 10^6$. The smooth curve is the asymptotic density function for $(2\sqrt{n} - L_n)/n^{1/6}$, based on theorem of Jinho Baik, Percy Deift, and Kurt Johansson. Data for the asymptotic distribution figure provided by Craig Tracy. Crosses represent the distribution of values of $(2\sqrt{n} - L_n)/n^{1/6}$ for $n = 10^5$ random permutations for $n = 10^6$.

A.M. Odlyzko, E.M. Rains, Preprint 1999

J. Baik, P. Deift, K. Johansson, math.CO/9810105