\[
p(\lambda) = \begin{cases} 
\frac{1}{\pi} \sqrt{\frac{2}{N}} \sqrt{1 - \frac{2\lambda^2}{2N}}, & 1 \leq (2N)^{1/2} \\
0 & \text{otherwise}
\end{cases}
\]

FIG. 1: The dashed line shows the semi-circular form of the average density of states. The largest eigenvalue is centered around its mean \(\sqrt{2N}\) and fluctuates over a scale of width \(N^{-1/6}\). The probability of fluctuations on this scale is described by the Tracy-Widom distribution (shown schematically).

scaling variable

\[\xi = \sqrt{2}N^{1/6} [\lambda_{\text{max}} - \sqrt{2N}]\]

\[\text{Prob} \left[ \xi \leq x \right] = F_\beta(x) \leftarrow \text{has a limit } N \to \infty \text{ (Tracy-Widom)}\]
The Marčenko–Pastur form of the average density of states for $c = 1$ is $\rho(\lambda) = N \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1}{x}} \mathrm{e}^{-\frac{1}{x^2}/2} \left[ 0, 4 \right]$. The average eigenvalue for $c = 1$ is $\langle \lambda \rangle = N$. For $c = 1$, the largest eigenvalue is centred around its mean $\langle \lambda_{\text{max}} \rangle = 4N$ and fluctuates over a scale of width $N^{1/3}$. The probability of fluctuations on this scale is described by the Tracy–Widom distribution (shown schematically).

The scaling variable is 

$$\Xi = c^{-1/6} x^+ N^{-1/3} (\lambda_{\text{max}} - x^+ N)$$

$$x^+ = \left( \frac{1}{c} + 1 \right)^2, \quad c = \frac{N}{M}$$

$$\text{Prob} (\Xi \leq x) = F_\rho(x) \quad \text{has a large N limit} \quad \text{(Tracy–Widom)}$$

From P. Vivo, S. Majumdar & D. Bohigas, cond-mat/0704.371

\textbf{Johansson} \textbf{Johnston}
$F_1(s)^2 = \exp \left( -\int_s^0 q(x) \,dx \right) \cdot F_2(s)$

$F_2(s) = \exp \left( -\int_s^0 (x-s) \,q(x) \,dx \right)$

$F_4(s/F_2)^2 = \cosh^2 \left( \frac{1}{2} \int_s^0 q(x) \,dx \right) \cdot F_2(s)$

where $q$ satisfies the Painlevé II equation

$q'' = xq + 2q^3$

with boundary condition $q(x) \sim \text{Ai}(x)$ as $x \to \infty$

$f_\beta(s) = \frac{dF_\beta(s)}{ds}$

$F_2(s) \to 1 - O(\exp[-4x^{3/2}]) \quad x \to \infty$

$\to \exp[-x^{3/12}] \quad x \to -\infty$
Figure 1: Densities for the scaled largest eigenvalues, $f_\beta(s)$.
Sequence of \( n \) distinct integers

\[ X = \{8, 3, 5, 4, 2, 1, 6, 7\} \quad n = 8 \]

subsequence: an ordered sublist of \( X : \{3, 1, 2, 6\} \)

longest increasing subsequence
\[ \{3, 5, 6, 7\}, \{1, 2, 6, 7\}, \{1, 2, 4, 7\} \]

length of the LIS \( l = 4 \)

\[ X = \{3, 7, 4, 5, 1, 2, 6, 8\} \]

LIS \( \{3, 4, 5, 6, 8\} \quad l = 5 \)

Consider all \( n! \) permutations equally likely (uniform measure).

For each permutation, find \( l_n \).

\( l_n \) is now a random variable.

What is the statistics of \( l_n \)? (Ullam's problem, 1961)

Ullam's conjecture \( \langle l_n \rangle \approx a \sqrt{n} \)

\( a \approx 2 \) ('68)

\( a = 2 \) (Vershik, 1977)
Longest increasing subsequence in a
random permutation of \{1, \ldots, n\}

\[ L_n \]

For large \( n \)

\[ \langle L_n \rangle = 2 \sqrt{n} \]

Distribution of

\[ \frac{2\sqrt{n} - L_n}{n^{1/6}} \]

is Tracy–Widom !! (\( \beta = 2 \))
length of the longest increasing subsequence of a random permutation of \( \{1, \ldots, n^3\} \)

\[ \langle L_n \rangle \sim 2 \sqrt{n} \quad \text{for } n \text{ large} \]

detailed statistics on increasing subsequences

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Figure 1: Asymptotic density function for \( n = 10^6 \). The smooth curve is the asymptotic density function for \( \frac{(2\sqrt{n} - L_n)}{n^{1/6}} \) based on theorem of Jimbo Baik, Percy Deift, and Kurt Johansson. Data for the asymptotic distribution figure provided by Craig Tracy. Crosses represent the distribution of values of \( \frac{(2\sqrt{n} - L_n)}{n^{1/6}} \) for \( n = 10^2 \) random permutations for \( n = 10^6 \).

A.M. Odlyzko, E.M. Rains, Preprint 1999
J. Baik, P. Deift, K. Johansson, math.CO/9810105