Large deviations and random matrices

Gaussian
Wishart } distribution of λ_{max}

given by Tracy-Widom for typical fluctuations

scale of typical fluctuations

$O(N^{-1/6})$ $\quad$ $\sqrt{2N}$ $\quad$ Gaussian

around mean

$O(N^{1/3})$ $\quad$ $x_+(c)\,N$ $\quad$ Wishart

Study of atypical and large fluctuations of λ_{max} around its mean, over a range

$O(N^{1/2})$ for Gaussian

$O(N)$ for Wishart

Question

$$\mathrm{Prob}\{\lambda_{max} < t\} = \Pr\{\lambda_1 < t, \lambda_2 < t, \ldots; \lambda_N < t\} = Q_N(t)$$

Find the equilibrium charge density of the Coulomb gas with a barrier at t

Restricted partition function

$$Z_N(t) = \int_{I(t)} \prod_{i=1}^{N} d\lambda_i \exp\left[-\frac{\beta}{2}\left(\sum_{i=1}^{N} V(\lambda_i) - \sum_{i<j} \ln(|\lambda_i - \lambda_j|)\right)\right]$$

$I(t)$: allowed range for eigenvalues

$(-\infty, t)$ Gaussian

$[0, t]$ Wishart

$$Q_N(t) = \frac{Z_N(t)}{Z_N(\infty)}$$

$V(x) = x^2$ Gaussian

$V(x) = x - [(1+M-N) - \frac{2}{\beta}] \log x$ Wishart

Rescaled variables

$$\mu = \lambda N^{-\alpha} \qquad \alpha = \tfrac{1}{2} \quad \text{Gaussian WD}$$

location of the barrier $\longrightarrow$

$$z = t N^{-\alpha} \qquad \alpha = 1 \quad \text{Wishart}$$

$$Z_N(z) \propto \int \mathcal{D}[\hat{f}] \exp\{ \beta N^2 S[\hat{f}(\mu; z] + O(N)\}$$

$$S[\hat{f}(\mu, z)] = -\frac{1}{2}\int_{I(z)} d\mu\, \hat{f}(\mu)\, V(\mu)$$

$$+ \frac{1}{2}\int_{I(z)}\int_{I(z)} d\mu\, d\mu'\, \hat{f}(\mu)\hat{f}(\mu') \ln|\mu - \mu'|$$

$I(z)$: allowed range of eigenvalues

Leading contribution to action, from

$$\frac{\delta S}{\delta f} = 0$$

with solution $\hat{f}$

Leads to equation of the form

$$P\int_0^z \frac{f(x')}{x-x'}\,dx' = g(x)$$

Its general solution (Tricomi)

$$f(x) = \frac{1}{\pi^2\sqrt{x(z-x)}}\left[P\int_0^z \sqrt{w(z-w)}\,\frac{g(w)}{w-x}\,dw + B\right]$$

WD Gaussian $\quad g(w) = w$

Wishart $\quad g(w) = \frac{1}{2}$

For Wishart ($c=1$)

$$\hat{f}(\mu) = \frac{1}{2\pi\sqrt{\mu(z-\mu)}}\left[\frac{z}{2} - 2 - \mu\right]$$

$$0 \leq \mu \leq z$$

From $\hat{f}(\mu)$ one can calculate the saddle point action

$$S[\hat{f}(\mu); z] \quad ,$$

and the restricted partition function

$$Z_N(z) \approx \exp\{\beta N^2 S(z)\}$$

constrained spectral density

$$\hat{\rho}_N(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda-\lambda_i)\,\theta(t-\lambda)$$

when an infinite barrier at t

The deformed semi-circle

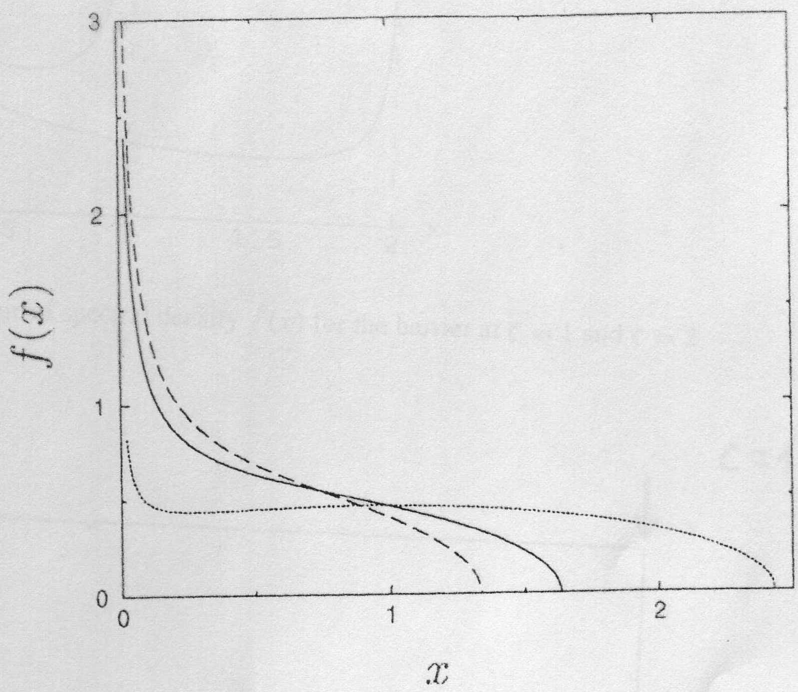

FIG. 2. The average density of states $f(x)$ plotted as a function of the shifted variable x for $z = -1$ (dotted line), $z = 0$ (solid line), and $z = 0.5$ (dashed line).

D. S. Dean, S. N. Majumdar, Phys. Rev. Lett. 97 (2006) 16020