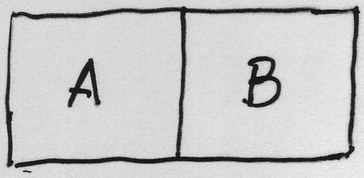
Entanglement of a random pure state

In quantum information and quantum computation

entanglement of states

Here, bipartite entanglement



dimension: N $\quad M$ $\qquad N \leq M$

basis $|i^A\rangle$ $\quad |\alpha^B\rangle$

$\mathcal{H}_A^{(N)}$ $\quad \mathcal{H}_B^{(M)}$

Hilbert space of composite system

$$\mathcal{H}^{(NM)} = \mathcal{H}_A^{(N)} \otimes \mathcal{H}_B^{(M)}$$

Any quantum state of the composite system

$$|\Psi\rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{M} x_{i,\alpha} \, |i^A\rangle \otimes |\alpha^B\rangle$$

$|i^A\rangle$: complete basis of $\mathcal{H}_A^{(N)}$

$|\alpha^B\rangle$: complete basis of $\mathcal{H}_B^{(M)}$

$x_{i,\alpha}$: entries of a rectangular $(N \times M)$ matrix $\mathbf{X}$

Properties (mutually non exclusive) of $|\Psi\rangle$

- entangled

fully unentangled when $x_{i,\alpha} = a_i b_\alpha$ for all i and α

$\therefore$ $|\Phi^A\rangle = \sum_{i=1}^{N} a_i |i^A\rangle$

$\Rightarrow |\Psi\rangle = |\Phi^A\rangle \otimes |\Phi^B\rangle$

$|\Phi^B\rangle = \sum_{\alpha=1}^{M} b_\alpha |\alpha^B\rangle$

- random

$x_{i,\alpha}$: random variables

- pure

if density matrix of composite system

$$\rho = |\Psi\rangle\langle\Psi|$$

$\mathrm{Tr}\,\rho = 1$, from $\langle\Psi|\Psi\rangle = 1$

((for composite system

$$\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$$

$0 \leq p_k \leq 1$

$|\Psi_k\rangle$ are pure states of the composite system

$0 \leq p_k \leq 1$ $\sum_k p_k = 1$))

density matrix of pure state $|\Psi\rangle$

$$\rho = \sum_{i,\alpha} \sum_{j,\beta} x_{i,\alpha} x^*_{j,\beta} |i^A\rangle\langle j^A| \otimes |\alpha^B\rangle\langle\beta^B|$$

reduced density matrix of subsystem A

$$\rho_A = \mathrm{Tr}_B\, \rho = \sum_{i,j=1}^{N} \sum_{\alpha=1}^{M} x_{i,\alpha} x^*_{j,\alpha} |i^A\rangle\langle j^A|$$

$$= \sum_{i,j=1}^{N} W_{i,j} |i^A\rangle\langle j^A|$$

W_{ij} : entries of $N \times N$ square matrix W

$$W = X X^+$$

W : eigenvalues $\lambda_1, \dots, \lambda_N$; $\lambda_i \geq 0$; $\sum_1^N \lambda_i = 1$

eigenvectors $|\lambda_i^A\rangle$

In this diagonal representation

$$\rho_A = \sum_{i=1}^{N} \lambda_i |\lambda_i^A\rangle\langle\lambda_i^A|$$

$$|\Psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i}\, |\lambda_i^A\rangle \otimes |\lambda_i^B\rangle$$

X^+X

Useful measure of entanglement

von Neumann entropy

$$S = -\sum_{i=1}^{N} \lambda_i \log \lambda_i$$

$\lambda_{max} = \frac{1}{N}$
all other λ's $= \frac{1}{N}$
$|\Psi\rangle$ maximally entangled

$\lambda_{max} = 1$
all other λ's $= 0$
$|\Psi\rangle$ maximally unentangled

λ_{max}

λ_{min}

$\lambda_{min} = 0$

dimensional reduction

$\lambda_{min} = \frac{1}{N}$

all other λ's $= \frac{1}{N}$

$|\Psi\rangle$ maximally entangled

Proximity of λ_{min} to borders provides information on entanglement and dimensional reduction