# Anderson Localization from Classical Trajectories

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Support: NSF,
Packard Foundation

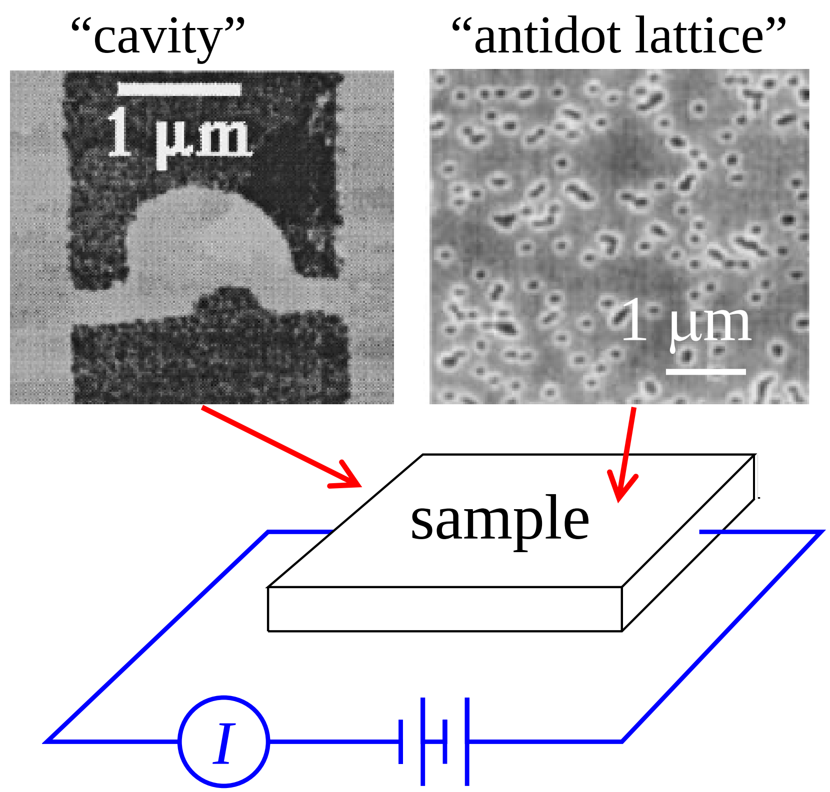
# Quantum Transport

Manifestations of the wave nature of electrons in electrical transport

- shot noise
- weak localization
- conductance fluctuations

    …

- Anderson localization




Originally discovered for disordered conductors.
This talk: ballistic conductors

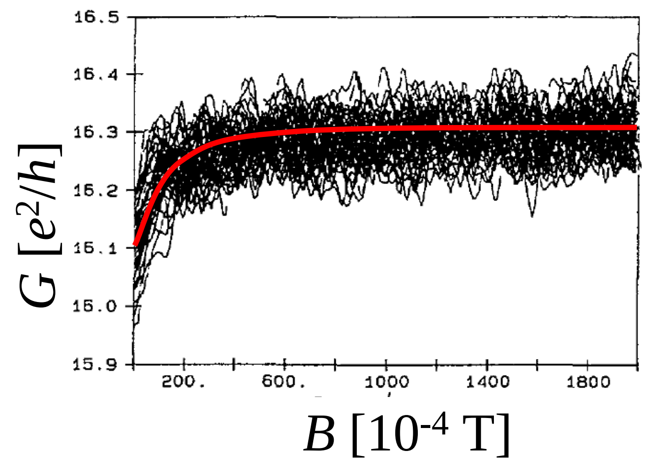
no scattering off
point-like impurities

# Weak localization

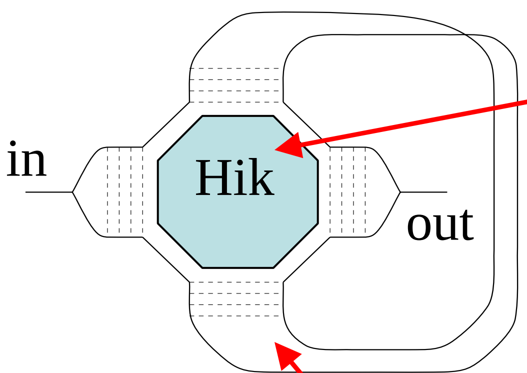
## disordered metals

Nonzero (negative) ensemble average
$\langle \delta G \rangle$ at zero magnetic field

$$G = \sum_{\mu} |A_{\mu}|^2 + \underbrace{\sum_{\mu \neq \nu} A_{\mu} A_{\nu}^{*}}_{\delta G}$$

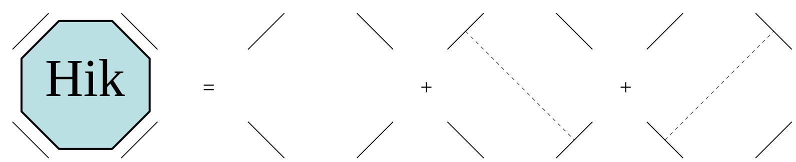



Mailly and Sanquer (1991)




“Hikami box”

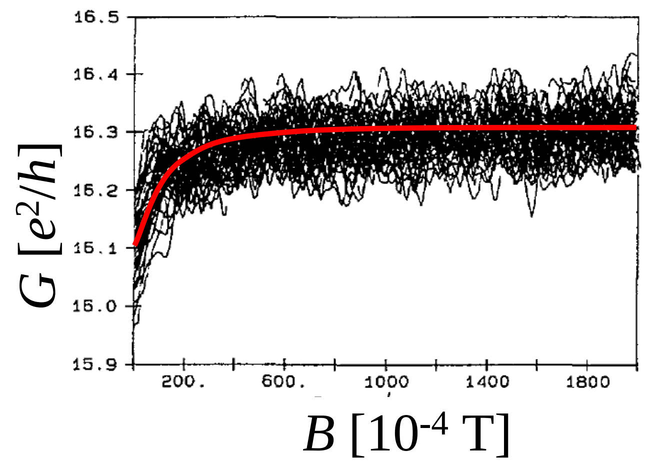
“Cooperon”



# Weak localization

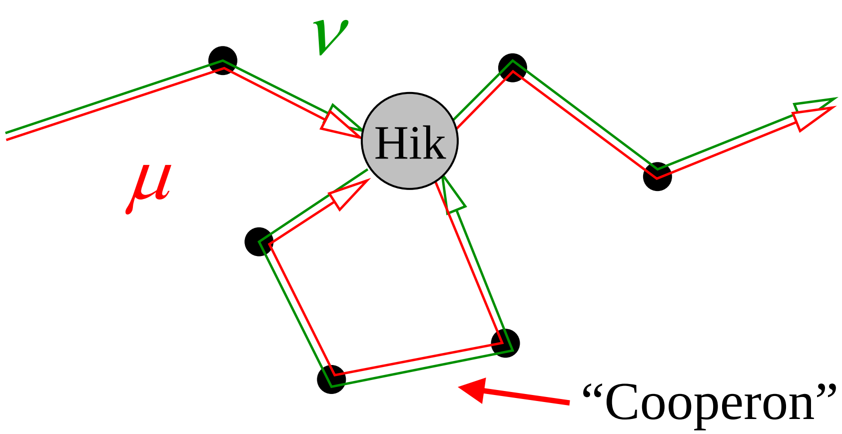
## disordered metals

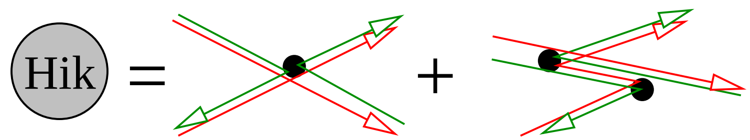
Nonzero (negative) ensemble average $\langle \delta G \rangle$ at zero magnetic field




$$G = \sum_{\mu} |A_{\mu}|^2 + \boxed{\sum_{\mu \neq \nu} A_{\mu} A_{\nu}^*}$$

$\delta G$



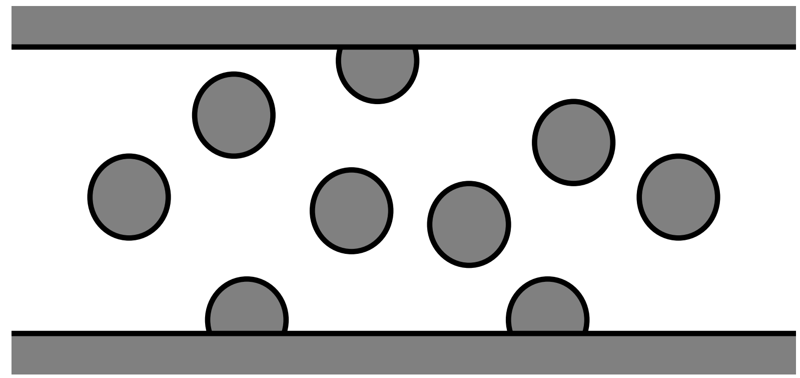

+ permutations

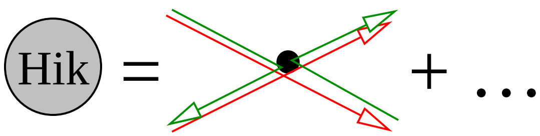
'Hikami box'

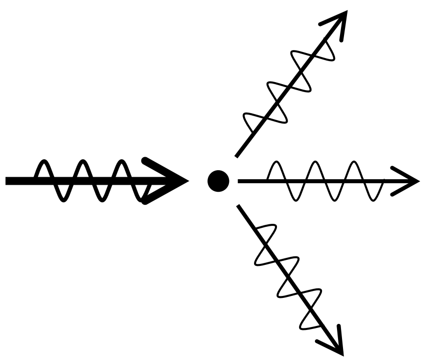
# Weak localization

## ballistic conductors

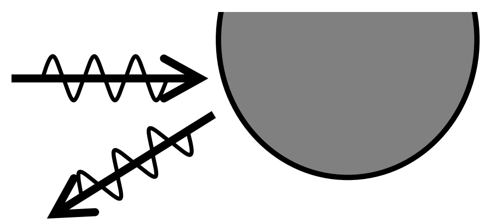
- Theory based on diffractive scattering off point-like impurities not possible;





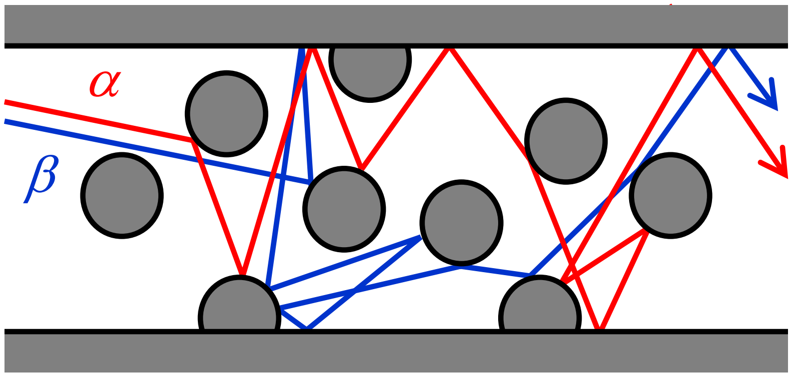
“disordered”



“ballistic”

# Weak localization
## ballistic conductors

- Theory based on diffractive scattering off point-like impurities not possible;




Instead: Semiclassics

$$g = \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar},$$

- $\alpha$ and $\beta$ have equal angles upon entrance/exit
- $S_{\alpha,\beta}$: classical action
- $A_{\alpha,\beta}$: stability amplitudes

Needed: Careful summation over classical trajectories $\alpha$, $\beta$.


Jalabert, Baranger, Stone (1990)
Argaman (1995)
Aleiner, Larkin (1996)
Richter, Sieber (2001,2002)
Heusler, Müller, Braun, Haake (2006)

# Weak localization

## ballistic conductors

$$g \sim \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar},$$

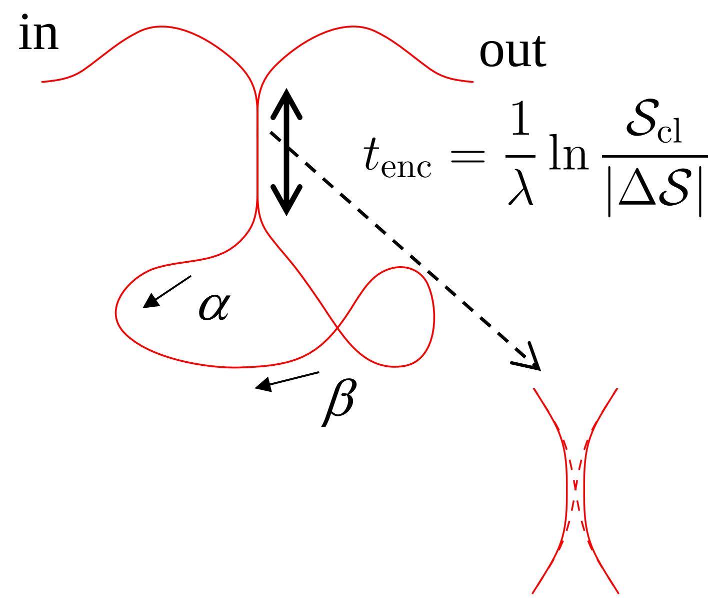
Weak localization: Trajectory pairs with small-angle self encounter

Sieber, Richter (2001)
also: Aleiner, Larkin (1996)

Encounter duration $t_{\mathrm{enc}} = \tau_{\mathrm{E}} = \frac{1}{\lambda} \ln \frac{\mathcal{S}_{\mathrm{cl}}}{\hbar}$

If $\tau_{\mathrm{E}} \ll$ dwell time: Recover weak localization correction of disordered metal

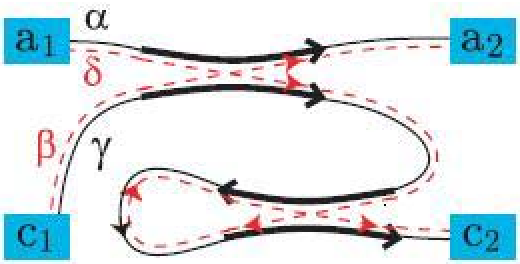
Aleiner, Larkin (1996)
Richter, Sieber (2002)
Heusler *et al*. (2006)
Brouwer (2007)




“ballistic Hikami box”

# Beyond weak localization
## ballistic conductors

$$g \sim \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar},$$




Braun et al. (2006)

One or more small-angle self encounters

- shot noise
- conductance fluctuations
- quantum pump
- full counting statistics
- time delay
- …

Braun *et al.* (2006)
Whitney and Jacquod (2006)
Brouwer and Rahav (2006)
Rahav and Brouwer (2006)
Berkolaiko *et al.* (2007)
Kuipers and Sieber (2007)
…

If $\tau_E$ << dwell time: Recover quantum corrections of disordered metals

# Beyond weak localization
## ballistic conductors

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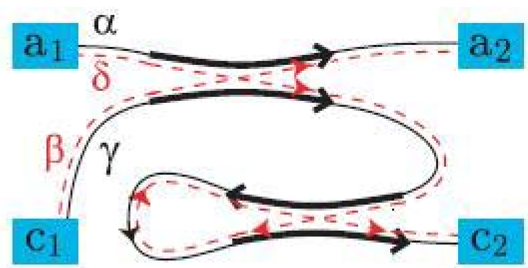
One or more small-angle self encounter

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Kuipers and Sieber (2007)

If $\tau_E$ << dwell time: Recover quantum corrections of disordered metals

But all of these are perturbative effects!

# Non-perturbative effects

<u>Level correlations:</u>

Form factor $K(t)$ for $|t| > \tau_H$

Heusler, Müller, Altland, Braun, Haake (2007)

"inspired by field theoretical formulation of RMT correlation functions"

Heusler et al. (2007)

# Non-perturbative effects

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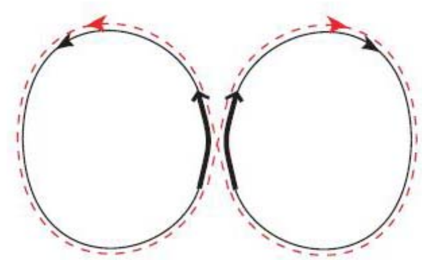
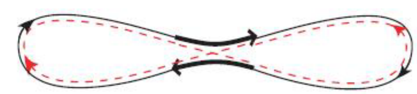


"inspired by field theoretical formulation of RMT correlation functions"



Heusler et al. (2007)

Today: Anderson localization

… inspired by theory of Anderson localization in disordered metals

- one-dimensional nonlinear sigma model

Efetov and Larkin (1983)

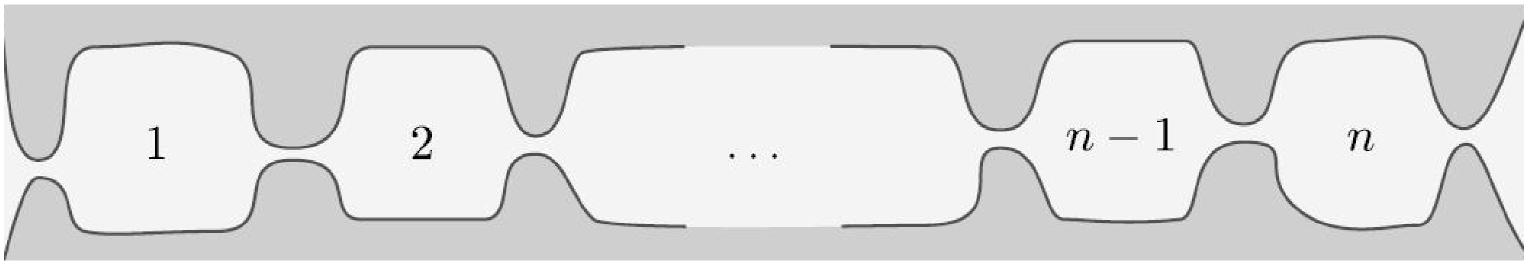
- scaling approach

Dorokhov (1982)

Mello, Pereyra, Kumar (1988)

# Anderson localization

## disordered metals




Model system: array of "quantum dots"

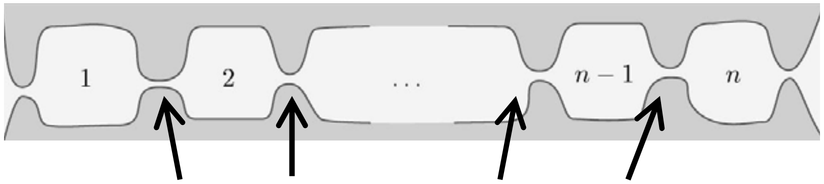
Dots are connected via ballistic contacts with conductance $g_c >> 1$.
Take limit $g_c \rightarrow \infty$ while keeping ratio $g_c/n$ fixed.

Disordered quantum dots: random matrix theory

Localization in quantum dot array:
Mirlin, Müller-Groeling, Zirnbauer (1994)
Brouwer, Frahm (1996)

# Anderson localization
## disordered metals

$$S(n) = \begin{pmatrix} S_{11}(n) & S_{12}(n) \\ S_{21}(n) & S_{22}(n) \end{pmatrix}$$




$$\mathcal{T}(n) = S_{12}(n) S_{12}^\dagger(n)$$

$$T_m(n) = \mathrm{tr}\, \mathcal{T}(n)^m$$

$g(n) = T_1(n)$ : conductance of array of *n* dots

random matrix theory: recursion relation for moments of the $T_i$:

$$\begin{aligned} \delta\langle T_1\rangle &= \langle T_1(n)\rangle - \langle T_1(n-1)\rangle \\ &= -\frac{1}{g_c}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_c^{-2}) \end{aligned}$$

(no time-reversal symmetry, $\beta$=2)

Replace difference equation by differential equation:

$$\frac{\partial}{\partial L}\langle T_1\rangle = -\frac{2}{\xi}\langle T_1^2\rangle$$

$L/\xi = n/2g_c$ $\xi$: "localization length"

# Anderson localization

## disordered metals

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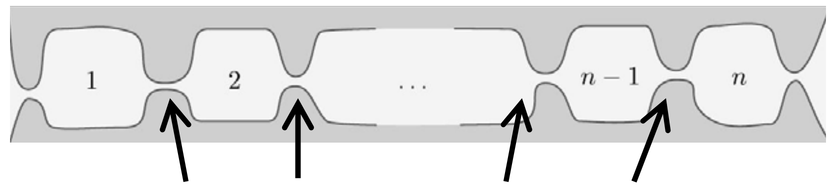



interdot conductance: $g_{\mathrm{c}}$

$$\mathcal{T}(n) = S_{12}(n) S_{12}^{\dagger}(n)$$

$$T_m(n) = \operatorname{tr} \mathcal{T}(n)^m$$

general recursion relation:

$$\begin{aligned} \delta \left\langle \prod_{m=1}^{n} T_{i_m} \right\rangle = & -\frac{1}{g_{\mathrm{c}}} \left( \sum_{k=1}^{n} i_k \right) \left\langle T_1 \prod_{m=1}^{n} T_{i_m} \right\rangle \\ & + \frac{1}{g_{\mathrm{c}}} \sum_{k=1}^{n} \sum_{j=1}^{i_k - 1} i_k \left\langle \left( T_j (T_{i_k - j} - T_{i_k - j + 1}) \right) \prod_{\substack{m=1 \\ m \neq k}}^{n} T_{i_m} \right\rangle \\ & + \frac{2}{g_{\mathrm{c}}} \sum_{k=1}^{n} \sum_{l=1}^{k-1} i_k i_l \left\langle \left( T_{i_k + i_l} - T_{i_k + i_l + 1} \right) \prod_{\substack{m=1 \\ m \neq k,l}}^{n} T_{i_m} \right\rangle + \mathcal{O}(g_{\mathrm{c}}^{-2}) \end{aligned}$$

# Anderson localization

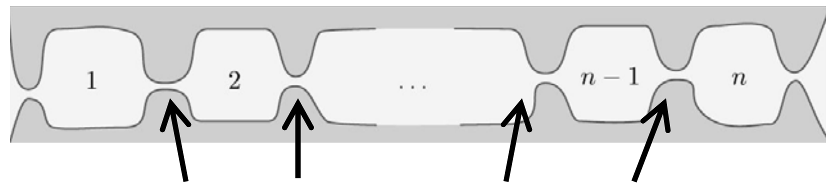
## disordered metals

$$S(n) = \begin{pmatrix} S_{11}(n) & S_{12}(n) \\ S_{21}(n) & S_{22}(n) \end{pmatrix}$$

$$\mathcal{T}(n) = S_{12}(n) S_{12}^{\dagger}(n)$$

$$T_m(n) = \mathrm{tr}\, \mathcal{T}(n)^m$$

$$\begin{aligned}
\delta \left\langle \prod_{m=1}^{n} T_{i_m} \right\rangle = & -\frac{1}{g_c} \left( \sum_{k=1}^{n} i_k \right) \left\langle T_1 \prod_{m=1}^{n} T_{i_m} \right\rangle \\
& + \frac{1}{g_c} \sum_{k=1}^{n} \sum_{j=1}^{i_k - 1} i_k \left\langle (T_j (T_{i_k - j} - T_{i_k - j + 1})) \prod_{\substack{m=1 \\ m \neq k}}^{n} T_{i_m} \right\rangle \\
& + \frac{2}{g_c} \sum_{k=1}^{n} \sum_{l=1}^{k-1} i_k i_l \left\langle (T_{i_k + i_l} - T_{i_k + i_l + 1}) \prod_{m=1}^{n} T_{i_m} \right\rangle + \mathcal{O}(g_c^{-2})
\end{aligned}$$




interdot conductance: $g_c$

Transform into differential equation for generating function:

$$F_2(\theta_1, \theta_3) = \left\langle \det \left( \frac{2 + (\cos(\theta_3) - 1)\mathcal{T}}{2 + (\cosh(\theta_1) - 1)\mathcal{T}} \right) \right\rangle$$

$$\frac{\partial}{\partial L} F_2 = \frac{2}{\xi} \sum_{j=1,3} \frac{1}{J(\theta_1, \theta_3)} \frac{\partial}{\partial \theta_j} J(\theta_1, \theta_3) \frac{\partial}{\partial \theta_j} F_2,$$

$$J(\theta_1, \theta_3) = \frac{\sin(\theta_3) \sinh(\theta_1)}{(\cosh(\theta_1) - \cos(\theta_3))^2}.$$

Description equivalent to existing theory of localization in quantum wires

Efetov and Larkin (1983)
Dorokhov (1982)
Mello, Pereyra, Kumar (1988)

# Anderson localization

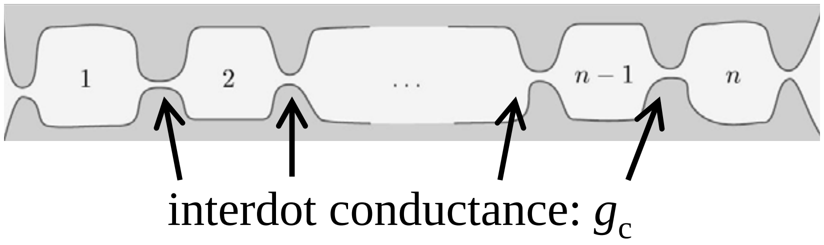
## disordered metals

$$S(n) = \begin{pmatrix} S_{11}(n) & S_{12}(n) \\ S_{21}(n) & S_{22}(n) \end{pmatrix}$$

$$\mathcal{T}(n) = S_{12}(n) S_{12}^{\dagger}(n)$$

$$T_m(n) = \mathrm{tr}\, \mathcal{T}(n)^m$$

$$\begin{aligned} \delta \left\langle \prod_{m=1}^{n} T_{i_m} \right\rangle = & -\frac{1}{g_c} \left( \sum_{k=1}^{n} i_k \right) \left\langle T_1 \prod_{m=1}^{n} T_{i_m} \right\rangle \\ & + \frac{1}{g_c} \sum_{k=1}^{n} \sum_{j=1}^{i_k - 1} i_k \left\langle (T_j (T_{i_k - j} - T_{i_k - j + 1})) \prod_{\substack{m=1 \\ m \neq k}}^{n} T_{i_m} \right\rangle \\ & + \frac{2}{g_c} \sum_{k=1}^{n} \sum_{l=1}^{k-1} i_k i_l \left\langle (T_{i_k + i_l} - T_{i_k + i_l + 1}) \prod_{m=1}^{n} T_{i_m} \right\rangle + \mathcal{O}(g_c^{-2}) \end{aligned}$$

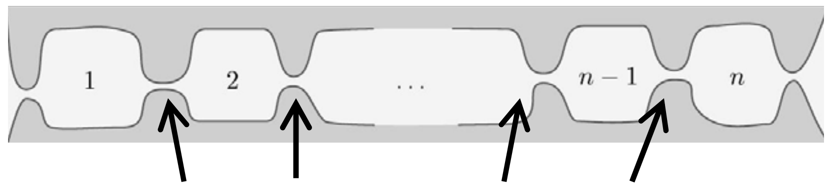



Can one derive the same set of recursion relations from semiclassics?

# Anderson localization

## ballistic conductors

$$S(n) = \begin{pmatrix} S_{11}(n) & S_{12}(n) \\ S_{21}(n) & S_{22}(n) \end{pmatrix}$$




$$\mathcal{T}(n) = S_{12}(n) S_{12}^{\dagger}(n)$$

$$T_m(n) = \mathrm{tr}\, \mathcal{T}(n)^m$$

Can we show that
$$\begin{aligned} \delta\langle T_1\rangle &= \langle T_1(n)\rangle - \langle T_1(n-1)\rangle \\ &= -\frac{1}{g_c}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_c^{-2}) \end{aligned}$$

from semiclassical expression for $T_1$?

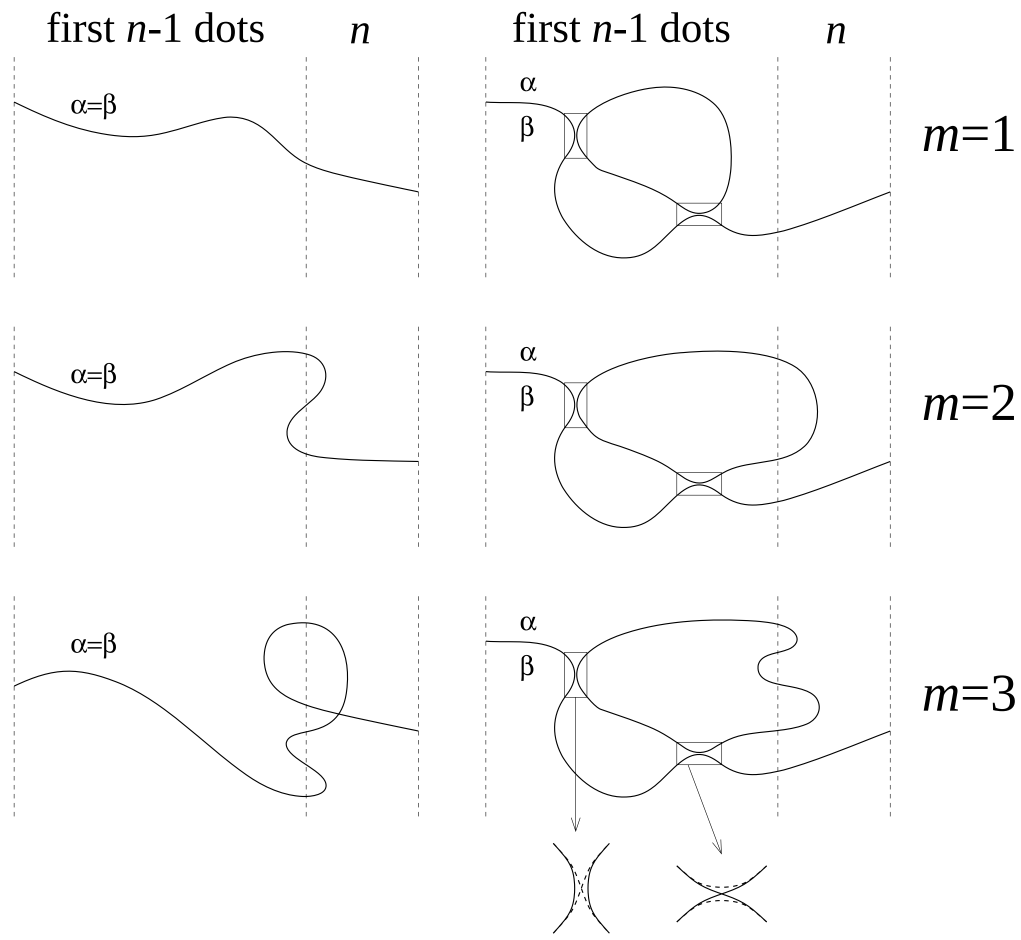
$$T_1 = \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar}$$

# Anderson localization

## ballistic conductors

$$T_1 = \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar}$$

- $\alpha$ and $\beta$ each have $m$ segments in $n^{\text{th}}$ dot, $\alpha_1,\ldots,\alpha_m;\ \beta_1,\ldots,\beta_m$.



# Anderson localization

## ballistic conductors

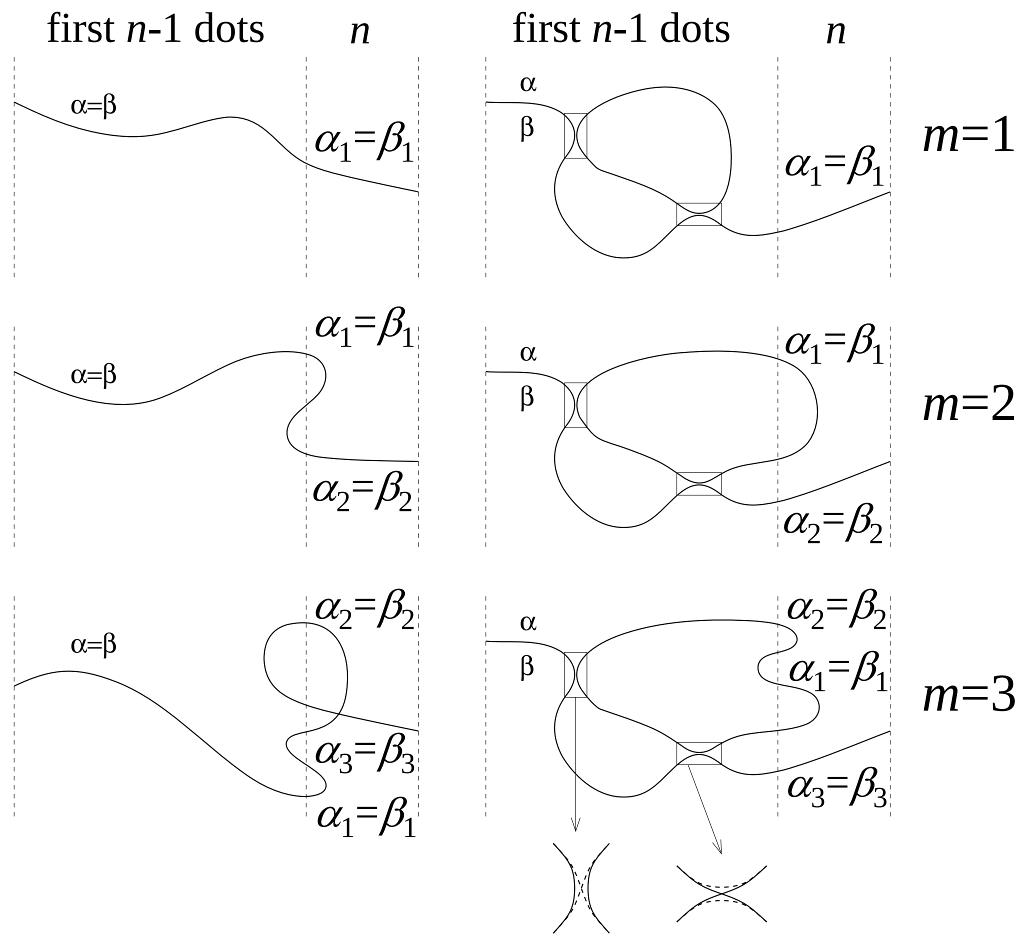
$$T_1 = \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar}$$

- $\alpha$ and $\beta$ each have $m$ segments in $n^{\text{th}}$ dot, $\alpha_1,\ldots,\alpha_m$; $\beta_1,\ldots,\beta_m$.

To leading order in $g_c$:

- diagonal approximation in $n^{\text{th}}$ dot
- pair $\alpha_i$ with $\beta_i$, $i$=1,…,$m$

No restriction on number of small-angle self encounters in first $n$-1 dots



# Anderson localization

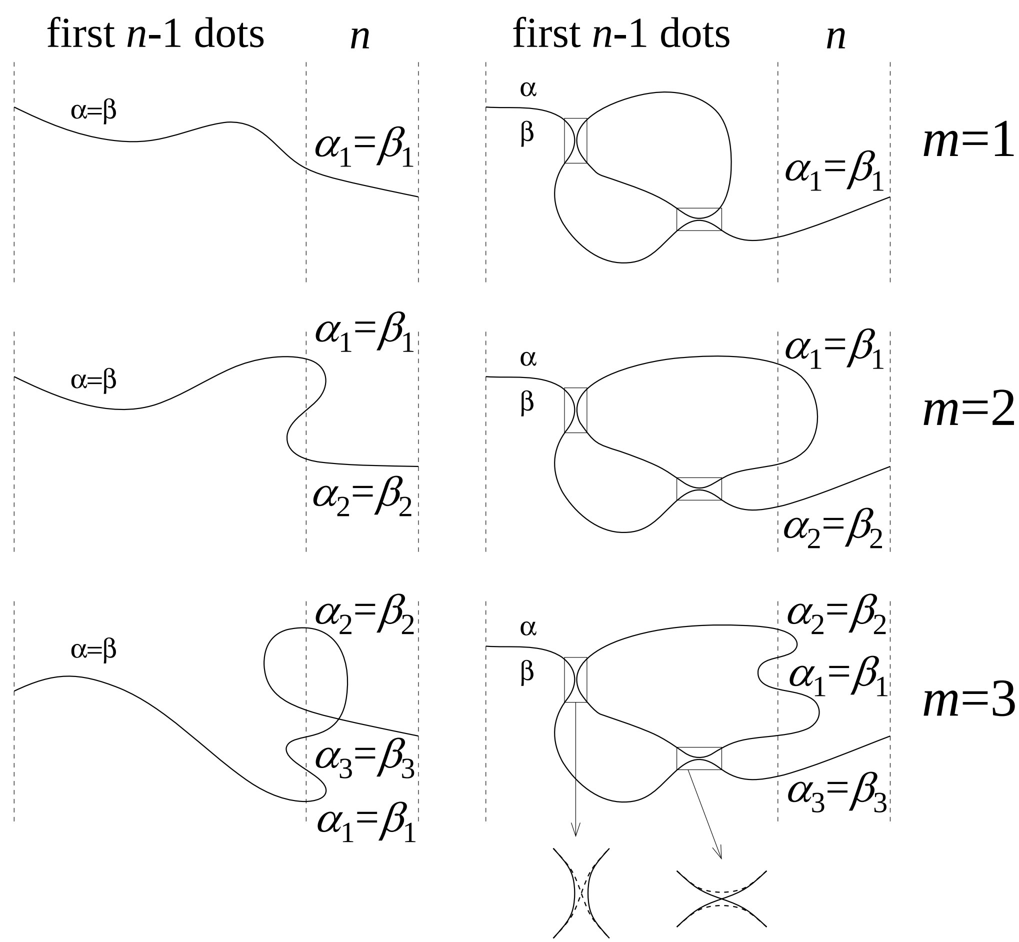
## ballistic conductors

$$\frac{1}{2}\langle T_1(n-1)\rangle$$

$$\frac{1}{4g_\mathrm{c}}\langle T_1(n-1)R_1(n-1)\rangle$$

**reflection:**
$R_1 = g_\mathrm{c} - T_1$

$$\frac{1}{8g_\mathrm{c}^2}\langle T_1(n-1)R_1(n-1)^2\rangle$$



# Anderson localization

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$$\cdots \qquad +$$

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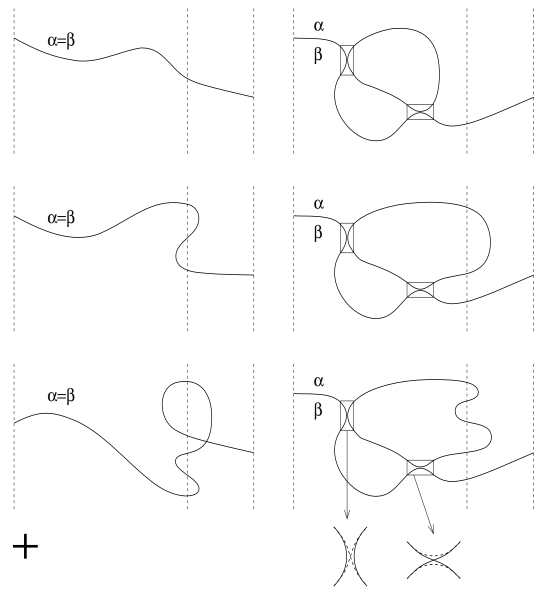



*m*=1

*m*=2

*m*=3

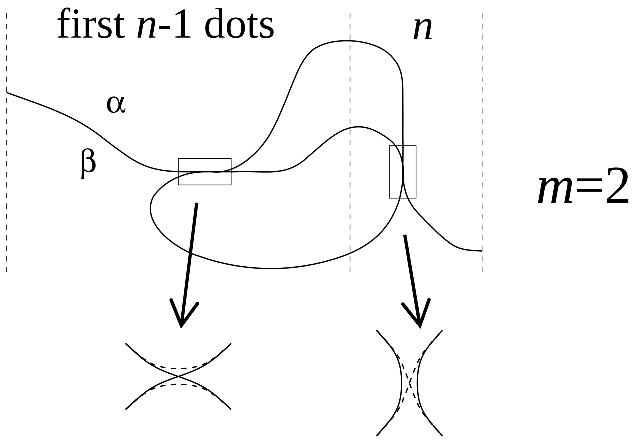
$$\begin{aligned}\langle T_1(n)\rangle &= \sum_{m=1}^{\infty} \frac{1}{2^m g_\mathrm{c}^{m-1}} \left\langle T_1(n-1)R_1(n-1)^{m-1}\right\rangle \\ &= \left\langle \frac{g_\mathrm{c}T_1(n-1)}{2g_\mathrm{c} - R_1(n-1)} \right\rangle \\ &= \langle T_1(n-1)\rangle - \frac{1}{g_\mathrm{c}}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_\mathrm{c}^{-2})\end{aligned}$$

# Anderson localization
## ballistic conductors

Beyond diagonal approximation in $n^{\text{th}}$ dot:

• contribution of order $g_c^{-2}$

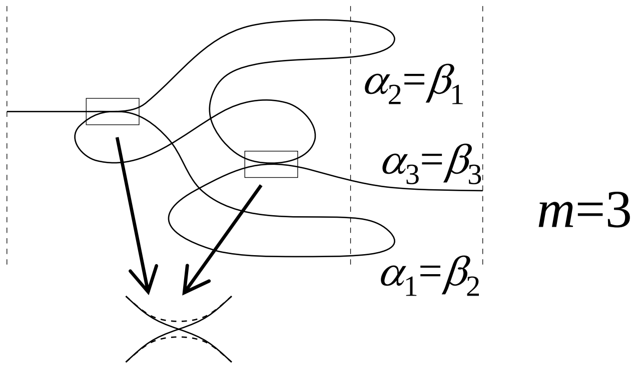
$$-\frac{1}{8g_c^2}\langle \mathrm{tr}\, S_{12}S_{22}^\dagger S_{22}S_{12}^\dagger\rangle$$
$$= \frac{1}{8g_c^2}\langle T_2(n-1) - T_1(n-1)\rangle$$




Pairing $\alpha_i$ with $\beta_j$, $i \neq j$:

• contribution of order $g_c^{-2}$

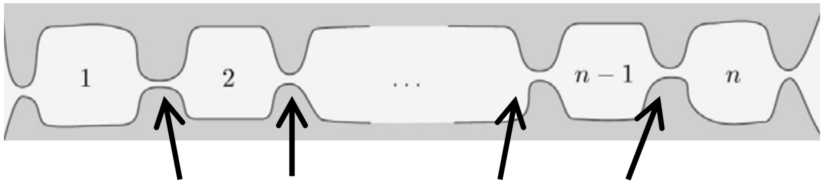
$$\frac{1}{8g_c^2}\langle \mathrm{tr}\, S_{12}(S_{22}^\dagger S_{22})^2 S_{12}^\dagger\rangle$$
$$= \frac{1}{8g_c^2}\langle T_1(n-1) - 2T_2(n-1) + T_3(n-1)\rangle$$



# Anderson localization

## ballistic conductors

Summarizing…




$$\delta\langle T_1\rangle = \langle T_1(n)\rangle - \langle T_1(n-1)\rangle$$
$$= -\frac{1}{g_c}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_c^{-2})$$

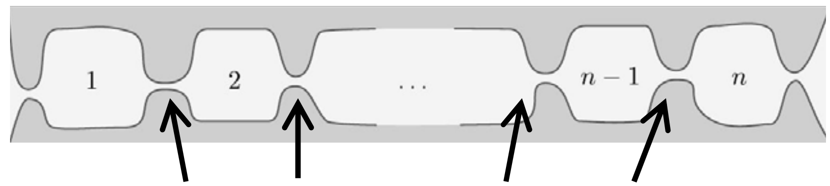
Extension to higher moments or $\beta$=1 (time-reversal symmetry):

- Need to consider up to one encounter in $n^{\text{th}}$ dot;
- Need to go (slightly) beyond pairing $\alpha_i$ with $\beta_i$, $i$=1,…,$m$.

# Anderson localization

## ballistic conductors

Summarizing…




$$\begin{aligned}\delta\langle T_1\rangle &= \langle T_1(n)\rangle - \langle T_1(n-1)\rangle \\ &= -\frac{1}{g_\mathrm{c}}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_\mathrm{c}^{-2})\end{aligned}$$

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…   …

# Anderson localization

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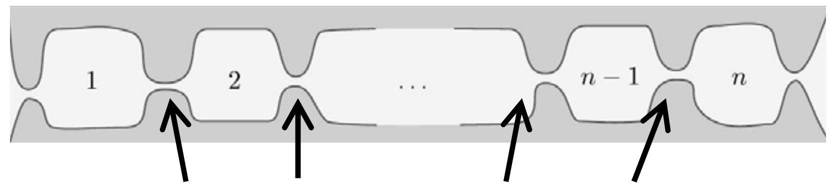



interdot conductance: $g_c$

<u>Summarizing…</u>

$$\begin{aligned}\delta\langle T_1\rangle &= \langle T_1(n)\rangle - \langle T_1(n-1)\rangle \\ &= -\frac{1}{g_\mathrm{c}}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_\mathrm{c}^{-2})\end{aligned}$$

Extension to higher moments and $\beta$=1 (time-reversal symmetry):

- Need to consider up to one encounter in $n^{\mathrm{th}}$ dot;
- Need to go (slightly) beyond pairing $\alpha_i$ with $\beta_i$, $i$=1,…,$m$.

… …

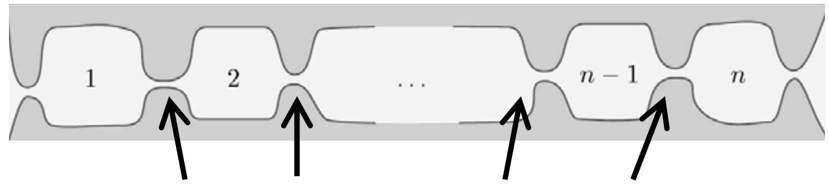
full set of recursion relations

$$\begin{aligned}\delta\left\langle\prod_{m=1}^{n} T_{i_m}\right\rangle =& -\frac{1}{g_\mathrm{c}}\delta_{\beta,1}\sum_{k=1}^{n} i_k\left\langle T_{i_k+1}\prod_{\substack{m=1\\ m\neq k}}^{n} T_{i_m}\right\rangle - \frac{1}{g_\mathrm{c}}\left(\sum_{k=1}^{n} i_k\right)\left\langle T_1\prod_{m=1}^{n} T_{i_m}\right\rangle \\ &+\frac{1}{g_\mathrm{c}}\sum_{k=1}^{n}\sum_{j=1}^{i_k-1} i_k\left\langle\left(T_j(T_{i_k-j}-T_{i_k-j+1})\right)\prod_{\substack{m=1\\ m\neq k}}^{n} T_{i_m}\right\rangle \\ &+\frac{4}{\beta g_\mathrm{c}}\sum_{k=1}^{n}\sum_{l=1}^{k-1} i_k i_l\left\langle\left(T_{i_k+i_l}-T_{i_k+i_l+1}\right)\prod_{\substack{m=1\\ m\neq k,l}}^{n} T_{i_m}\right\rangle + \mathcal{O}(g_\mathrm{c}^{-2}),\end{aligned}$$

# Anderson localization

## ballistic conductors

<u>Summarizing…</u>




interdot conductance: $g_{\mathrm{c}}$

$$\begin{aligned}\delta\langle T_1\rangle &= \langle T_1(n)\rangle - \langle T_1(n-1)\rangle \\ &= -\frac{1}{g_{\mathrm{c}}}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_{\mathrm{c}}^{-2})\end{aligned}$$

Theory of Anderson localization in array of ballistic chaotic cavities, formulated in terms of classical trajectories only.

Brouwer and Altland, arXiv:0802.0976

$$\delta\left\langle \prod_{m=1} T_{i_m}\right\rangle = -\frac{}{g_{\mathrm{c}}}\delta_{\beta,1}\sum_{k=1} i_k \left\langle T_{i_k+1}\prod_{\substack{m=1\\ m\neq k}} T_{i_m}\right\rangle - \frac{}{g_{\mathrm{c}}}\left(\sum_{k=1} i_k\right)\left\langle T_1 \prod_{m=1} T_{i_m}\right\rangle$$

$$+\frac{1}{g_{\mathrm{c}}}\sum_{k=1}^{n}\sum_{j=1}^{i_k-1} i_k \left\langle (T_j(T_{i_k-j}-T_{i_k-j+1}))\prod_{\substack{m=1\\ m\neq k}}^{n} T_{i_m}\right\rangle$$

$$+\frac{4}{\beta g_{\mathrm{c}}}\sum_{k=1}^{n}\sum_{l=1}^{k-1} i_k i_l \left\langle (T_{i_k+i_l}-T_{i_k+i_l+1})\prod_{\substack{m=1\\ m\neq k,l}}^{n} T_{i_m}\right\rangle + \mathcal{O}(g_{\mathrm{c}}^{-2}),$$