The effect of spin in the spectral statistics of quantum graphs

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Outline

Motivation

Bohigas-Gianonni-Schmit conjecture

Spectral statistics of quantized chaotic systems with time-reversal symmetry depend on the spin quantum no. s.

Bosons	integer s	COE statistics
Fermions	half-integer s	CSE statistics

- **Circular orthogonal ensemble** (COE)
  U symmetric unitary matrix.
  $\mu(U) = \mu(O^{-1}UO)$ for O orthogonal.
- **Circular symplectic ensemble** (CSE)
  U symplectic unitary matrix $U^T JU = J$, $J = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}$.
  $\mu(U) = \mu(S^{-1}US)$ for S symplectic.

Random matrix spectral statistics

U unitary matrix, eigenvalues $\mathrm{e}^{\mathrm{i}\phi_1}, \dots, \mathrm{e}^{\mathrm{i}\phi_N}$.

$$\rho(\phi) = \frac{1}{N} \sum_{i=1}^{N} \delta(\phi - \phi_i) = \frac{1}{2\pi N} \sum_{i=1}^{N} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n(\phi-\phi_i)}$$

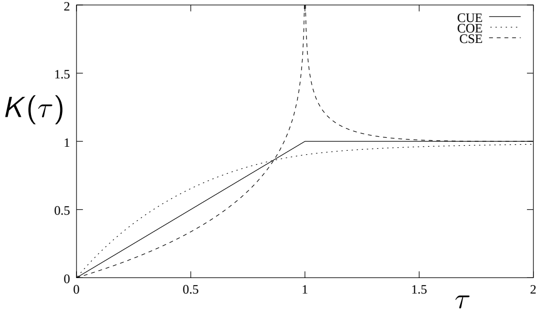
$$= \frac{1}{2\pi N} \sum_{n=-\infty}^{\infty} (\mathrm{tr}\, U^n) \mathrm{e}^{\mathrm{i}n\phi}$$

Two-point correlation function $R_2(\Delta\phi) = \langle \rho(\phi)\rho(\phi + \Delta\phi) \rangle$.

Definition (form factor)

fourier transform of R_2

$$K(\tau) = \frac{1}{N} \left\langle |\, \mathrm{tr}\, U^t |^2 \right\rangle \qquad \tau = \frac{t}{N}$$

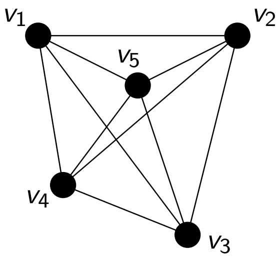


Power series expansion

$$K_{\mathrm{CSE}}(\tau) = \frac{\tau}{2} + \frac{\tau^2}{4} + \frac{\tau^3}{8} + \frac{\tau^4}{12} + \dots$$

$$\frac{1}{2}K_{\mathrm{COE}}\left(\frac{\tau}{2}\right) = \frac{\tau}{2} - \frac{\tau^2}{4} + \frac{\tau^3}{8} - \frac{\tau^4}{12} + \dots$$

Metric graph G



$\mathcal{V}$ set of vertices
$\mathcal{E}$ set of edges, $E = |\mathcal{E}|$.
$e = (u, v) \in \mathcal{E}$ if $u \sim v$.

Metric graph

Each edge e associated with interval $[0, L_e]$.

Free Pauli op. on G

Laplace op. $-\frac{\mathrm{d}^2}{\mathrm{d}x_e^2}$ on $L^2(G) \otimes \mathbb{C}^n$.
($n = 2s + 1$ *where* s *is the spin quantum no.*)

Matching conditions *(Kostrykin & Schrader)*

At vertex v valency k matching conditions defined by $nk \times nk$ matrices $\mathbb{A}_v, \mathbb{B}_v$

$$\mathbb{A}_v \boldsymbol{\psi}_v + \mathbb{B}_v \boldsymbol{\psi}'_v = \mathbf{0} \ .$$

The operator is self-adjoint iff $(\mathbb{A}_v, \mathbb{B}_v)$ maximal rank and $\mathbb{A}_v \mathbb{B}_v^\dagger = \mathbb{B}_v \mathbb{A}_v^\dagger$.

Vertex scattering matrices

Matching conditions chosen s.t. $\mathbb{S}_v = U_v^{-1}\big(\Sigma_v \otimes \mathrm{I}_n\big)U_v$.

$$U_v = \begin{pmatrix} R_n(u_1) & & \\ & \ddots & \\ & & R_n(u_k) \end{pmatrix}$$

where $u_j \in \Gamma \subseteq \mathrm{SU}(2)$ and $R_n(\Gamma)$ is an irrep. dim n.

Σ_v vertex scattering matrix of Laplace op. on $L^2(G)$.

$$\Sigma_v = -(\mathbb{A}'_v + \mathrm{i}k\mathbb{B}'_v)^{-1}(\mathbb{A}'_v - \mathrm{i}k\mathbb{B}'_v)$$

Then $\mathbb{A}_v = (\mathbb{A}'_v \otimes \mathrm{I}_n)U_v$ and $\mathbb{B}_v = (\mathbb{B}'_v \otimes \mathrm{I}_n)U_v$.

Time-reversal op. T_n, $T_n^2 = (-\mathrm{I})^{n+1}$.
$\mathbb{S}_v$ is **time-reversal symmetric** if $\Sigma_v^T = \Sigma_v$.

S-matrix ensemble

$$S_{\phi}^{(ij)(lm)} := \delta_{jl}\, \sigma_{(ij)(jm)}\, R_n(u^{(ij)(jm)})\, \mathrm{e}^{\mathrm{i}\phi_{\{ij\}}}$$

$u^{(ij)(jm)} = (u_i^{(j)})^{-1} u_m^{(j)}$ spin transformation $(ij) \to (jm)$,
$u^{(mj)(ji)} = (u^{(ij)(jm)})^{-1}$.

Trace formula

$$\mathrm{tr}\, S_{\phi}^{t} = \sum_{p \in P_t} \frac{t}{r_p} A_p\, \mathrm{e}^{\mathrm{i}\pi\mu_p}\, \chi_R(d_p)\, \mathrm{e}^{\mathrm{i}\phi_p}$$

$p = (e_1, e_2, \ldots, e_t)$ periodic orbit of G, $\chi_R(d) = \mathrm{tr}\,(R_n(d))$.

$$\begin{aligned} A_p\, \mathrm{e}^{\mathrm{i}\pi\mu_p} &:= \sigma_{e_1 e_2} \sigma_{e_2 e_3} \ldots \sigma_{e_{t-1} e_t} \\ d_p &:= u^{e_1 e_2} u^{e_2 e_3} \ldots u^{e_{t-1} e_t} \\ \phi_p &:= \phi_{e_1} + \cdots + \phi_{e_t} \end{aligned}$$

Graph form factor

$$K_{\mathrm{orth}/\mathrm{sym}}(\tau) := \frac{1}{2\times 2En} \left\langle |\operatorname{tr} S_\phi^t|^2 \right\rangle_\phi \ , \qquad \tau = \frac{2t}{2En}$$

$$= \frac{t^2}{2\times 2En} \sum_{p,q\in P_t} \frac{A_p A_q}{r_p r_q} \mathrm{e}^{\mathrm{i}\pi(\mu_p-\mu_q)} \chi_R(d_p)\chi_R^*(d_q)\, \delta_{\phi_p,\phi_q}$$

Kramers' degeneracy

If $T^2 = -\mathrm{I}$, half-integer spin (n even), eigenvalues of S are doubly degenerate.

Spin contribution to form factor

Diagram D

A set of orbit pairs related by the same pattern of permutation and or time reversal of arcs between self intersections.

$$K^{D}_{\mathrm{orth/sym}} := \frac{t^2}{2\times 2En} \sum_{(p,q)\in D_t} A_p A_q \, \mathrm{e}^{\mathrm{i}\pi(\mu_p-\mu_q)} \, \chi_R(d_p)\chi^*_R(d_q)$$

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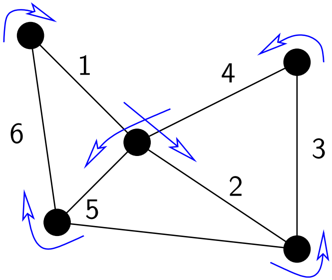
Assume d_p chosen randomly from $\Gamma \subseteq \mathrm{SU}(2)$.

$$K^{D}_{\mathrm{orth/sym}} = \frac{1}{2n} \left(\frac{1}{|D_t|} \sum_{(p,q)\in D_t} \chi_R(d_p)\chi_R^*(d_q) \right) \left(\frac{t^2}{2E} \sum_{(p,q)\in D_t} A_p A_q \, \mathrm{e}^{\mathrm{i}\pi(\mu_p-\mu_q)} \right)$$

In semiclassical limit

$$\frac{1}{|D_t|}\sum_{(p,q)\in D_t}\chi_R(d_p)\chi_R^*(d_q) \to \frac{1}{|\Gamma|^t}\sum_{u_1,\ldots,u_t\in\Gamma}\chi_R(d_p)\chi_R^*(d_q)$$



$$d_p = u_1 u_2 u_3 u_4 u_5 u_6$$
$$d_q = u_1 (u_2 u_3 u_4)^{-1} u_5 u_6$$

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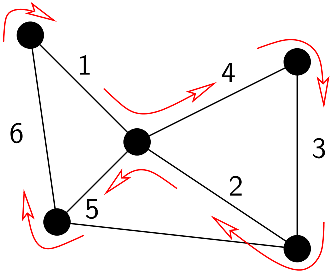

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Theorem

$$\frac{1}{|\Gamma|^t} \sum_{u_1,\dots,u_t \in \Gamma} \chi_R(d_p)\chi_R^*(d_q) = \left(\frac{c_R}{n}\right)^{m_D}$$

where m_D is the no. of self-intersections at which p was rearranged to produce q, $c_R = 1$ for real irreps and $c_R = -1$ for R quaternionic.

Idea of proof

Theorem

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Idea of proof

1. $$\sum_{xy \in \Gamma} \chi_R(xy)\chi_R^*(xy^{-1}) = \frac{c_R}{n} \sum_{xy \in \Gamma} \chi_R(xy)\chi_R^*(xy)$$

Theorem

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Idea of proof

1. $$\sum_{xy\in\Gamma}\chi_R(xy)\chi_R^*(xy^{-1})=\frac{c_R}{n}\sum_{xy\in\Gamma}\chi_R(xy)\chi_R^*(xy)$$
2. $$\sum_{xyz\in\Gamma}\chi_R(xyz)\chi_R^*(xzy)=\left(\frac{1}{n}\right)^2\sum_{xyz\in\Gamma}\chi_R(xyz)\chi_R^*(xyz)$$

First step

Lemma

$$\frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_R(ug^2) = \frac{c_R}{n}\chi_R(u)$$

Let $w = xy$

$$\begin{aligned}\sum_{xy\in\Gamma}\chi_R(xy)\chi_R^*(xy^{-1}) &= \sum_{wy\in\Gamma}\chi_R(w)\chi_R^*(wy^{-2}) \\ &= \frac{c_R}{n}|\Gamma|\sum_{w\in\Gamma}\chi_R(w)\chi_R^*(w) \\ &= \frac{c_R}{n}\sum_{xy\in\Gamma}\chi_R(xy)\chi_R^*(xy)\end{aligned}$$

Result

$$K^D_{\mathrm{orth/sym}} = \frac{1}{2n}\left(\frac{c_R}{n}\right)^{m_D} \frac{t^2}{2E} \sum_{(p,q)\in D_t} A_p A_q \, \mathrm{e}^{\mathrm{i}\pi(\mu_p - \mu_q)}$$

Diagram D contributes at order τ^{m_D+1}.

Orthogonal case

n odd, $c_R = 1$ and $\tau = t/2En$.

$$K^D_{\mathrm{orth}} = K^D_{\mathrm{zero}}$$

Result

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Diagram D contributes at order τ^{m_D+1}.

Symplectic case

n even and $\tau = 2t/2En$.

$$K^D_{\mathrm{sym}} = \left(\frac{c_R}{2}\right)^{m_D+2} K^D_{\mathrm{zero}}$$

Compare with random matrix theory

$$K_{\mathrm{CSE}}(\tau) = \sum_{m=1}^{\infty}\left(-\frac{1}{2}\right)^{m+1} K^m_{\mathrm{COE}}\,\tau^m, \qquad 0 < \tau \leqslant 2\ .$$

Conclusion

- Evaluated spin contribution to form factor of quantum graphs with random spin transformations.
- CSE or COE statistics depend on the repn of the group of spin transformations.
- Consistent with B-G-S conjecture if $R(\Gamma)$ quaternionic irrep for half-integer spin, e.g.

$$\Gamma = \{\pm \mathrm{I}, \pm\sigma_x, \pm\sigma_y, \pm\sigma_z\} \ .$$

J. Bolte and J. M. Harrison, *J. Phys. A* **36**, L433–L440 (2003). `arXiv nlin.CD/0304046`

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