Scarring on invariant manifolds for quantum maps on the torus

DUBI KELMER

Banff workshop

Quantum chaos: Routes to RMT and beyond

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Overview

- Quantum ergodicity and scarring
- Quantum maps on the torus
- Scarring on periodic orbits
- Scarring on invariant manifolds

Quantization

- Classical flow: $\Phi^t : X \to X$

- Unitary flow U^t on a Hilbert space $\mathcal{H}_h$

- Quantum states $\psi \in \mathcal{H}_h$ interpreted as distribution $\mathcal{W}_\psi$ on X.

- Semi-classical limit $h \to 0$ retrieve classical behavior ($\mathcal{W}_{U^t\psi} \sim \mathcal{W}_\psi \circ \phi^t$)

- If ψ is an eigenfunction then in semiclassical limit $\mathcal{W}_\psi$ becomes Φ^t invariant.

- What are the possible limiting (invariant) measures?

Quantum Ergodicity

- Chaotic dynamics: Small changes in initial condition result in drastic changes in outcome

- Lose all information as $t \to \infty$

- Quantum interpretation: for "small" h, $U^t\psi$ become evenly distributed as $t \to \infty$

- Eigenfunctions become evenly distributed as $h \to 0$

Quantum Ergodicity Theorem: When the classical dynamics is ergodic, in the semiclassical limit $\mathcal{W}_\psi \to \mathrm{vol}$ for "almost all" eigenfunctions

QUE and scarring

- Scarring: There are eigenfunction localizing on an invariant set

$$\lim_{h \to 0} \mathcal{W}_\psi(P) \neq 0$$

- Quantum Unique Ergodicity (QUE): the volume measure is the only limiting measure

- QUE is conjectured to hold for negatively curved surfaces. (Proved for "arithmetic" surfaces [Lindenstrauss])

Theorem (Anantharaman,Koch,Nonnenmacher). *For Anosov flows, any limiting measure has positive entropy.*

Conjecture. *Entropy bounded below by half of maximal entropy*

Quantum mechanics on the torus

- Phase space $\mathbb{T}^{2d} = \mathbb{R}^{2d}/\mathbb{Z}^{2d}$
  coordinates $x = \begin{pmatrix} p \\ q \end{pmatrix}$

- Hilbert space $\mathcal{H}_h = L^2[(h\mathbb{Z}/\mathbb{Z})^d]$
  (where $h = 1/N$).

- Weyl quantization:

$$f \in C^\infty(\mathbb{T}^{2d}) \rightsquigarrow \mathrm{Op}_h(f)$$

- Wigner distribution

$$\psi \in \mathcal{H}_h \rightsquigarrow \mathcal{W}_\psi(f) = \langle \mathrm{Op}_h(f)\psi, \psi \rangle$$

- If $f = f(q)$ (function of position)

$$\mathcal{W}_\psi(f) = h^d \sum f(\frac{Q}{N})|\psi(\frac{Q}{N})|^2$$

Quantization of Hamiltonian flows

- $H \in C^{\infty}(\mathbb{T}^{2d})$ real valued Hamiltonian

- Hamiltonian flow $\Phi_H^t : \mathbb{T}^{2d} \to \mathbb{T}^{2d}$

$$\frac{d}{dt}(f \circ \Phi_H^t) = \{f, H\} \circ \Phi_H^t$$

- Quantization: unitary flow

$$U(\phi_H^t) = \mathsf{exp}(\frac{it}{\hbar}\,\mathsf{Op}_h(H))$$

- Egorov Theorem:

$$U^* \,\mathsf{Op}_h(f) U = \mathsf{Op}_h(f \circ \Phi_H) + O(h)$$

implying:

$$\mathcal{W}_{U\psi}(f) = \mathcal{W}_{\psi}(f \circ \Phi_H) + O(h)$$

Quantization of linear maps

- $A \in \mathsf{Sp}(2d, \mathbb{Z})$ acts on $\mathbb{T}^{2d}$ ($x \mapsto Ax \pmod 1$)

- A hyperbolic implies map is Anosov

- Quantization [Hannay-Berry]: There is a unique unitary operator satisfying that

$$U_h(A)^* \mathsf{Op}_h(f) U_h(A) = \mathsf{Op}_h(f \circ A)$$

[Remark: $A \mapsto U(A)$ is the Weil representation of $\mathsf{Sp}(2d, \mathbb{Z}/N\mathbb{Z})$]

- For perturbation: $\Phi = \Phi_H \circ A$ quantization: $U(\Phi) = U(\Phi_h)U(A)$.

Scarring on periodic orbits

Theorem (Faure, Nonnenmacher, De-Bievre). *For linear maps on $\mathbb{T}^2$, there are e.f. satisfying*

$$\mathcal{W}_\psi(f) \to \frac{1}{2}f(0) + \frac{1}{2}\int f dx$$

- These scars occur only when $U(A)$ has large spectral degeneracies

- Arithmetic symmetries remove degeneracies

  With arithmetic symmetries linear maps on $\mathbb{T}^2$ are QUE [Kurlberg, Rudnick]

- Perturbation remove degeneracies

  Open question: Is a generic perturbation QUE?

Invariant manifolds for linear maps on $\mathbb{T}^{2d}$

- $A : \mathbb{T}^{2d} \to \mathbb{T}^{2d}$ by $x \mapsto Ax \pmod 1$

  Dual action: $A : \mathbb{Z}^{2d} \to \mathbb{Z}^{2d}$ by $n \mapsto nA$

- Correspondence: $\Lambda \subset \mathbb{Z}^{2d}$ invariant lattice of rank $d_0 \Rightarrow$

  $$X_\Lambda = \left\{ x \in \mathbb{T}^{2d} | e_n(x) = 1, \ \forall n \in \Lambda \right\}$$

  invariant manifold of co-dimension d_0.

- Invariant manifolds only exist for $d > 1$

- We say Λ is isotropic (or X_Λ co-isotropic) if the symplectic form

  $$\omega(n, m) = n_1 \cdot m_2 - n_2 \cdot m_1$$

  vanishes on $\Lambda \times \Lambda$.

Scarring on Invariant manifolds

Theorem (K.). *Let* $A \in \mathrm{Sp}(2d, \mathbb{Z})$*. Assume that* $\Lambda \subseteq \mathbb{Z}^{2d}$ *is invariant and isotropic:*

- *There are eigenfunctions of* $U(A)$ *localizing on* X_Λ

$$\mathcal{W}_\psi(f) \to \int_{X_\Lambda} f dx$$

- *This also holds after taking arithmetic symmetries into account*

- *This also holds for perturbation* $\Phi_H \circ A$ *by any Hamiltonian flow preserving* X_Λ

- *If* A *has a fixed point* $\xi \in \mathbb{T}^{2d}$ *there are also eigenfunctions localizing on* $X_\Lambda + \xi$.

The simplest example

- Take $A = \begin{pmatrix} B^t & 0 \\ 0 & B^{-1} \end{pmatrix}$ for $B \in \mathrm{GL}(d, \mathbb{Z})$

- Invariant manifold $X = \left\{ \begin{pmatrix} p \\ 0 \end{pmatrix} \in \mathbb{T}^{2d} \right\}$

- Perturbation: (by Hamiltonian $H = H(q)$)

$$\Phi(\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)) = \begin{pmatrix} B^t p + \nabla H(B^{-1}q) \\ B^{-1}q \end{pmatrix}$$

- Quantization:

$$U(\Phi)\psi(\frac{Q}{N}) = \exp(\frac{i}{\hbar} H(\frac{BQ}{N}))\psi(\frac{BQ}{N})$$

- The state $\psi = \delta_0$ is an eigenfunction of U localized on X

Sketch of proof

- Consider the family of operators

$$\mathcal{A} = \{\mathsf{Op}_h(e_n) | n \in \Lambda\}$$

- Λ isotropic $\Rightarrow \mathcal{A} \cong (\mathbb{Z}/N\mathbb{Z})^{d_0}$ is commutative.

- Decomposition into joint eigenspaces,

$$\mathcal{H}_h = \bigoplus \mathcal{H}_\lambda, \quad (\mathsf{Op}_h(e_n)\psi = \lambda(n)\psi).$$

  Each of dimension N^{d-d_0}.

- For any $\psi \in \mathcal{H}_1$ and $n \in \Lambda$, $\mathcal{W}_\psi(e_n) = \langle \mathsf{Op}_h(e_n)\psi, \psi \rangle = 1$.

- States from $\mathcal{H}_1$ are concentrated on X_Λ.

- Exact Egorov$\Rightarrow U(A)$ permutes eigenspaces

$$U(A) : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda \circ A}$$

- Hamiltonian flow preserves $X_\Lambda \Rightarrow$ $U(\Phi_H)$ preserves all eigenspaces

- The trivial eigenspace $\mathcal{H}_1$, is preserved by purerbed map: $U(\Phi_H \circ A)\mathcal{H}_1 = \mathcal{H}_1$.

- There is a basis for $\mathcal{H}_1$ composed of eigen-functions

- These are the localized eigenfunctions

Concluding remarks

- Quantum states localize on a co-isotropic manifold (uncertainty principle does not apply)

- The entropy of the scarred states is always bounded below by half the maximal entropy and it is equal if and only if $\dim X_\Lambda = d$

- If A has an invariant lattice then $U(A)$ has large spectral degeneracies.

  However, perturbation (generically) removes all degeneracies

- Scarring on invariant manifolds does not imply spectral degeneracies

THE END...