Constructing a sigma model from semiclassics

In collaboration with:
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Sebastian Müller

University of BRISTOL

Approaches to spectral statistics

Semiclassics

Approaches to spectral statistics

Semiclassics

Sigma model

Approaches to spectral statistics

Semiclassics

Sigma model

field-theoretical method for
averaging over random matrices, disorder

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doing combinatorics

Approaches to spectral statistics

Semiclassics

⇓

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doing combinatorics

⇓

Universal results, in agreement with RMT

Sigma model in RMT

Sigma model in RMT

Generating function

Sigma model in RMT

Generating function

$$Z = \left\langle \frac{\Delta(E+\epsilon_C)\,\Delta(E-\epsilon_D)}{\Delta(E+\epsilon_A)\,\Delta(E-\epsilon_B)} \right\rangle$$

$$\Delta(E) = det(E-H)$$

Sigma model in RMT

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- **replica** trick

$$\Delta(E) = \lim_{r\to 0} \Delta(E)^{-(r-1)}$$

Sigma model in RMT

random matrix average

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Gauss integrals

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Gauss integrals

⇓

traded for integral over matrices Q

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$$(2r \times 2r,\ \mathrm{Q}^2 = 1)$$

Sigma model in RMT

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$$Z = \int d[Q] \exp\left(\frac{1}{2}\mathrm{tr} X Q\right)$$

Sigma model in RMT

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energy differences

Relevance for semiclassics

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non-oscillatory / **oscillatory** terms:

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Z falls into integrals over two submanifolds

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$$Z \;=\; Z_1 \;+\; Z_2$$

non-oscillatory **oscillatory**

Relevance for semiclassics

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$$Z = Z_1 + Z_2$$

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same relation as with Riemann-Siegel !

(Berry & Keating, 1990;
Keating & S.M., 2007)

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(Berry & Keating, 1990;
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$$Z_2(\epsilon_A, \epsilon_B, \epsilon_C, \epsilon_D) = Z_1(\epsilon_A, \epsilon_B, -\epsilon_D, -\epsilon_C)$$

Relevance for semiclassics

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Analog of diagonal approximation:

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Analog of diagonal approximation:

rational parametrization

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$$Z_1 = \int d[B] \exp \left[\mathrm{tr} X' \left(\frac{1}{2} + \sum_{l=1}^{\infty} (BB^\dagger)^l \right) + \mathrm{tr} X'' \left(\frac{1}{2} + \sum_{l=1}^{\infty} (B^\dagger B)^l \right) \right]$$

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Analog of diagonal approximation:

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Keep only **Gaussian** terms!

Relevance for semiclassics

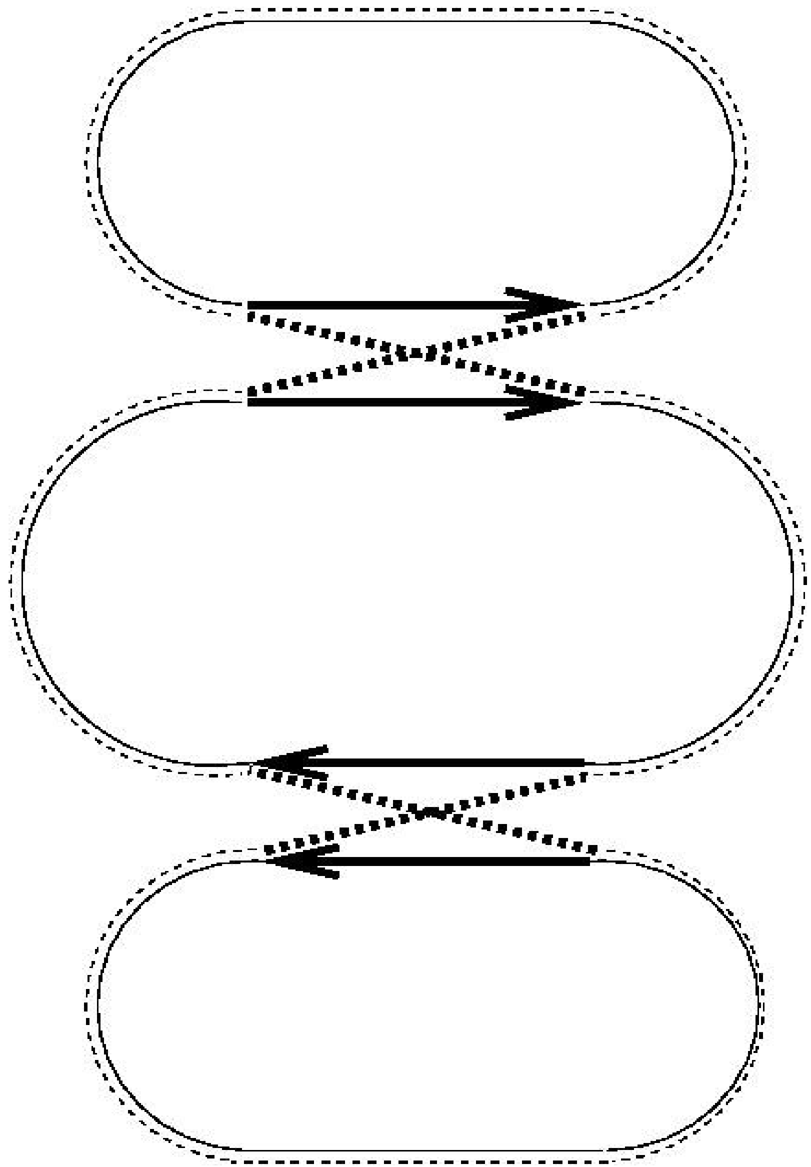
Keep all terms

➡ **perturbation theory**

Relevance for semiclassics

Keep all terms

➡ **perturbation theory**



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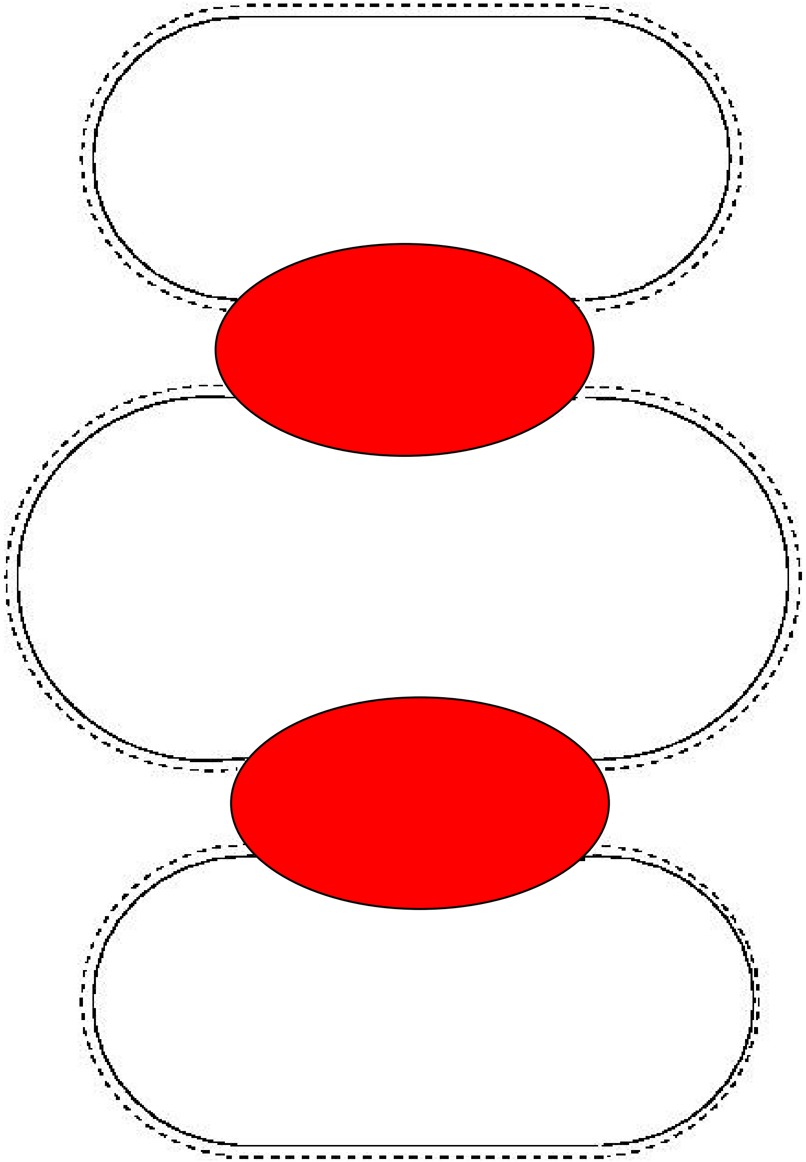

Relevance for semiclassics

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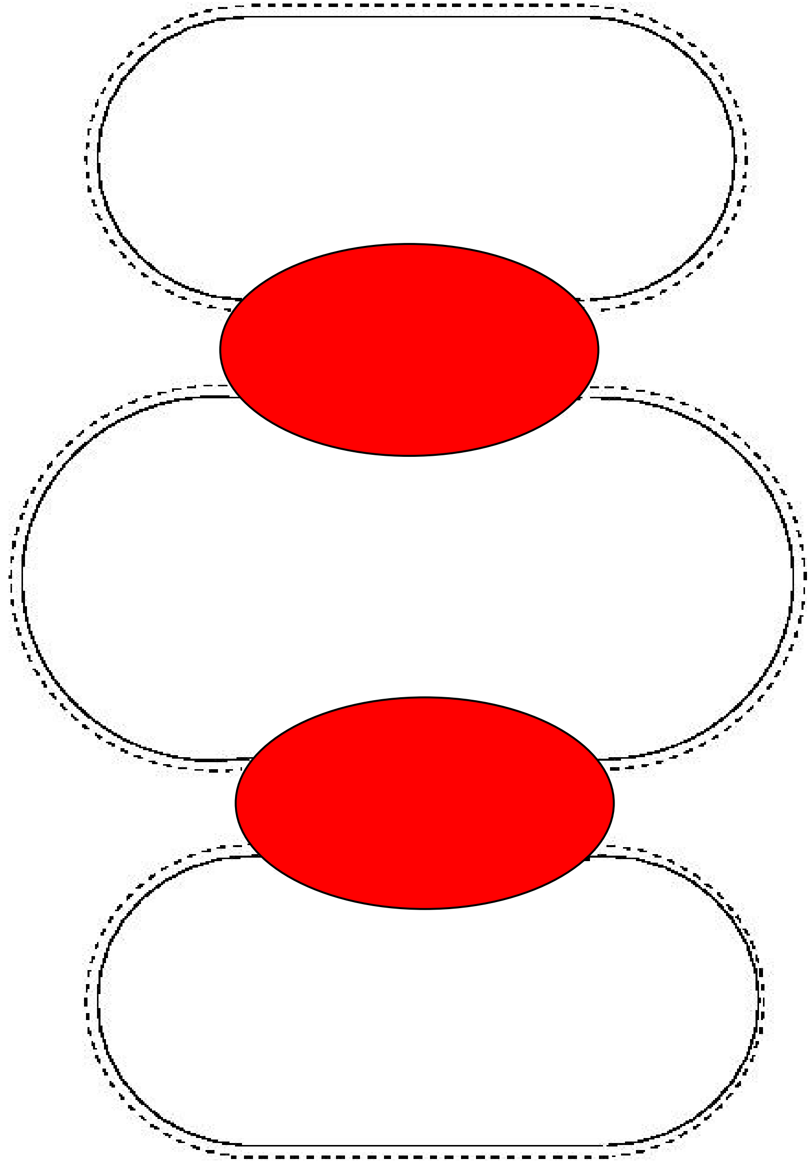
encounters =
vertices



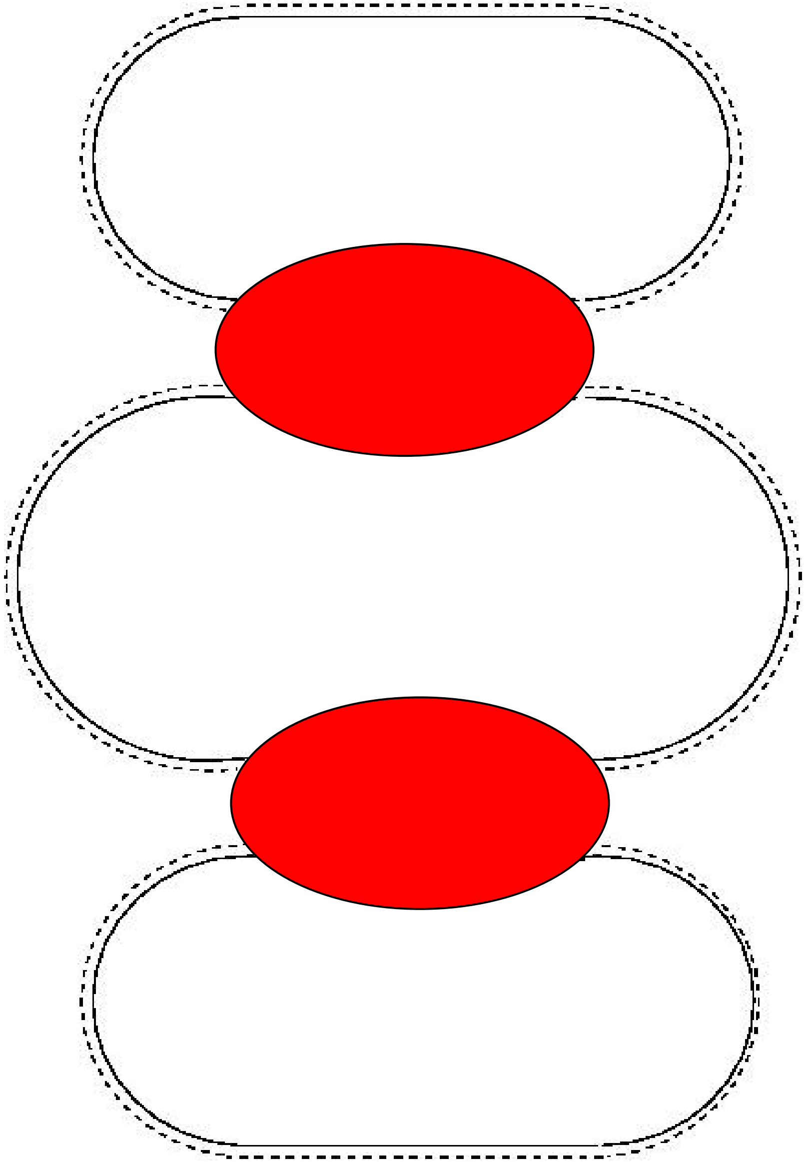
Relevance for semiclassics

Keep all terms

➡ **perturbation theory**

encounters =
vertices

links =
propagator lines



Constructing a sigma model from semiclassics

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Replica trick also works in semiclassics!

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r `original` pseudo-orbits

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Replica trick also works in semiclassics!

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differing in encounters

Drawing orbits

Drawing orbits

1 **Draw encounters**

Drawing orbits

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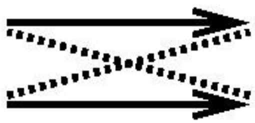
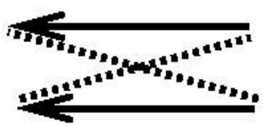
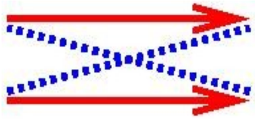
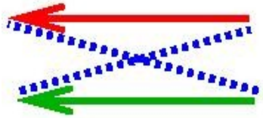




Drawing orbits

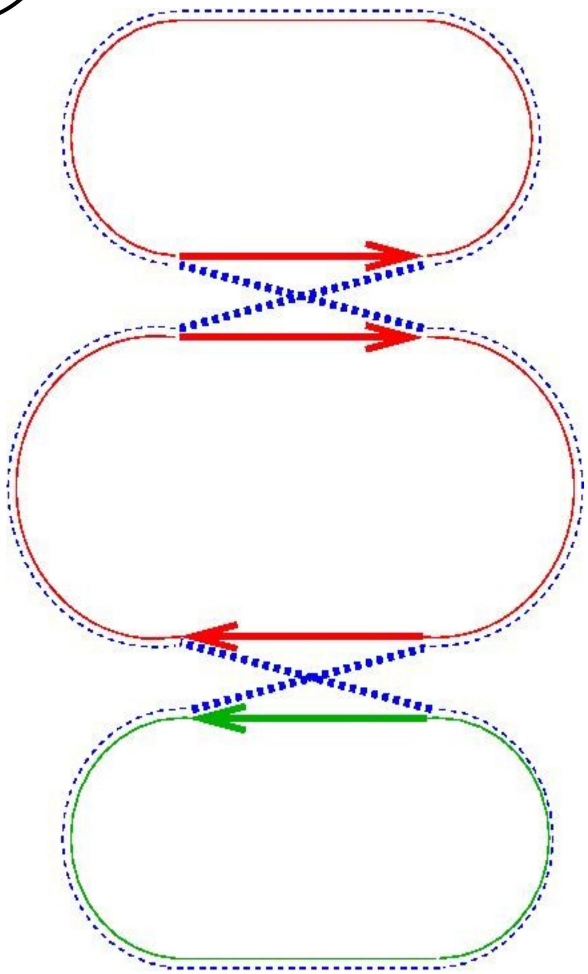
2 **Choose pseudo-orbits (colors)**





Drawing orbits

3 **Connect!**



Drawing orbits

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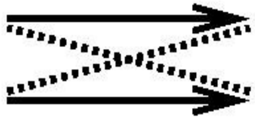

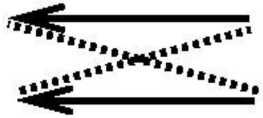

Drawing orbits

1 **Draw encounters**

write B for each entrance port,
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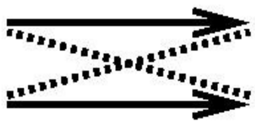

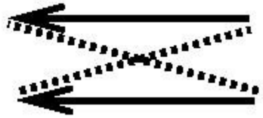

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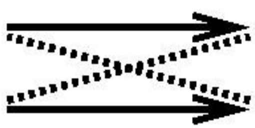
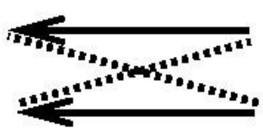


$B \quad B^\dagger \quad B \quad B^\dagger$



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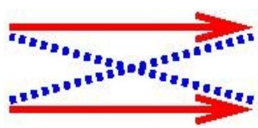
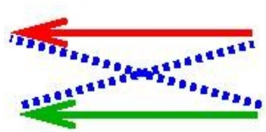
Drawing orbits

2 **Choose pseudo-orbit (colors)**



$$B \quad B^{\dagger} \quad B \quad B^{\dagger}$$



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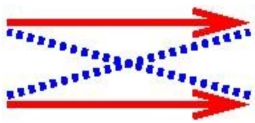
Drawing orbits

2 **Choose pseudo-orbit (colors)**

choose indices according to pseudo-orbits



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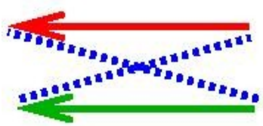

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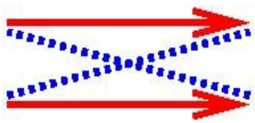

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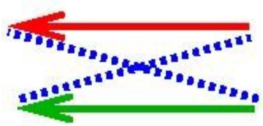

$$B_{31} B^{\dagger}_{13} B_{32} B^{\dagger}_{23}$$

Drawing orbits

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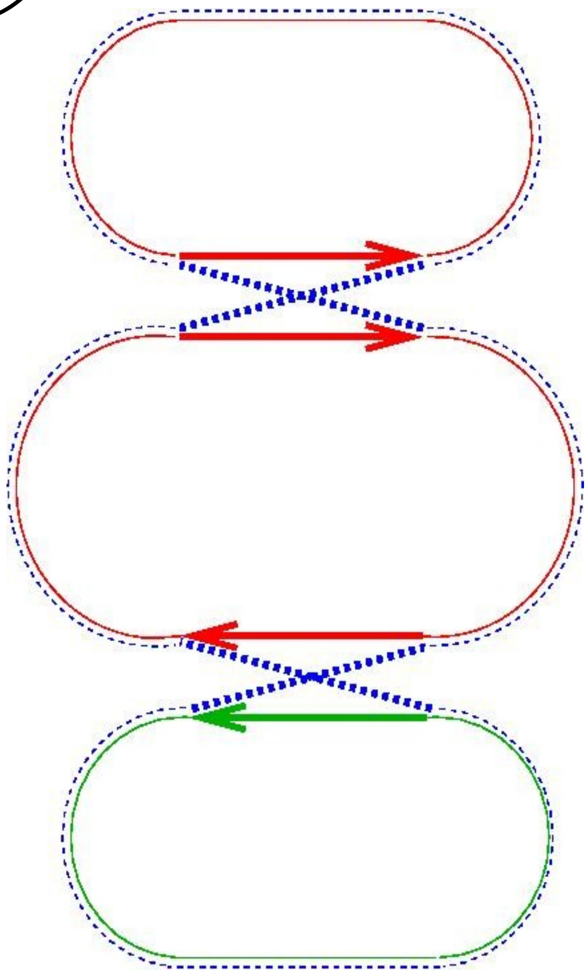

Drawing orbits

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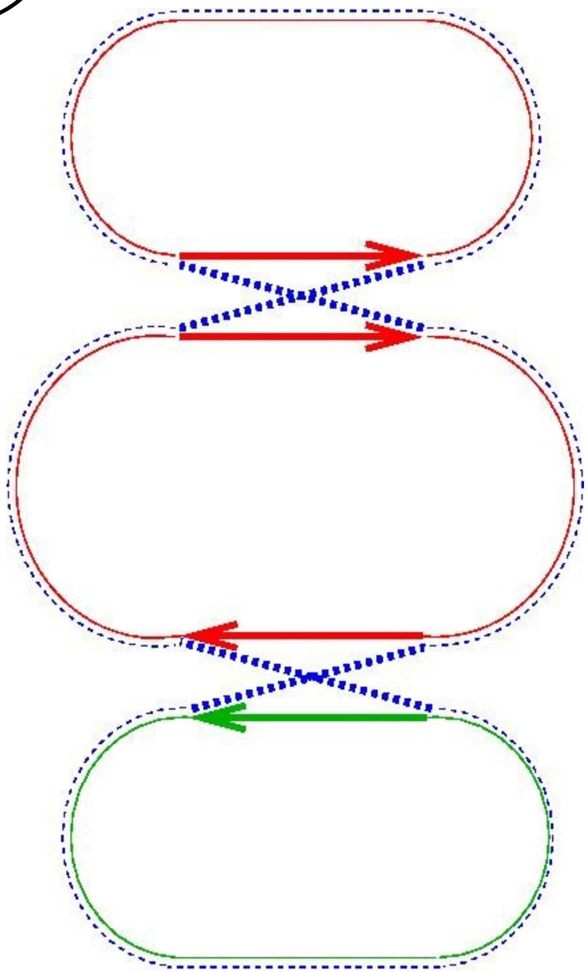

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Drawing orbits

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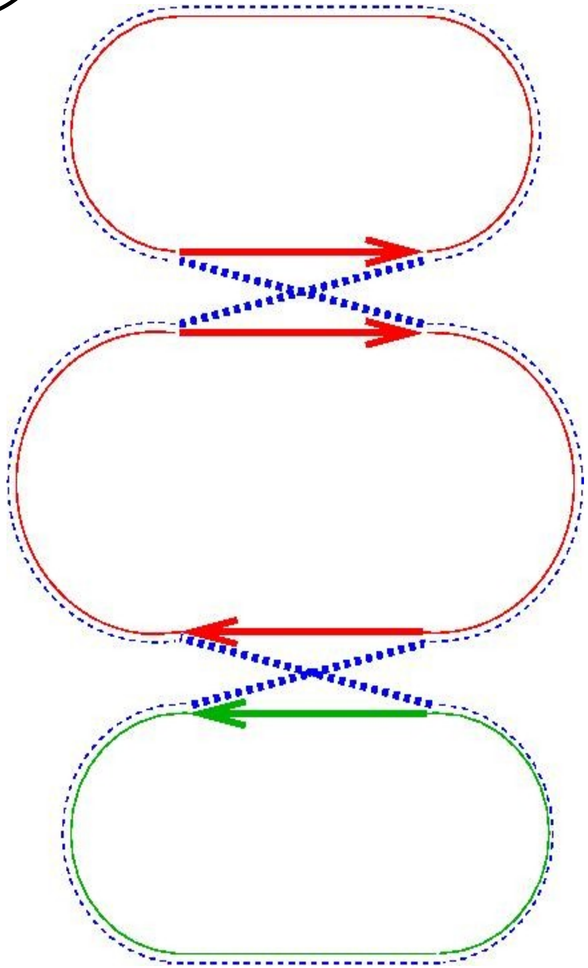

- numbers of entrances & corresp. exits must coincide
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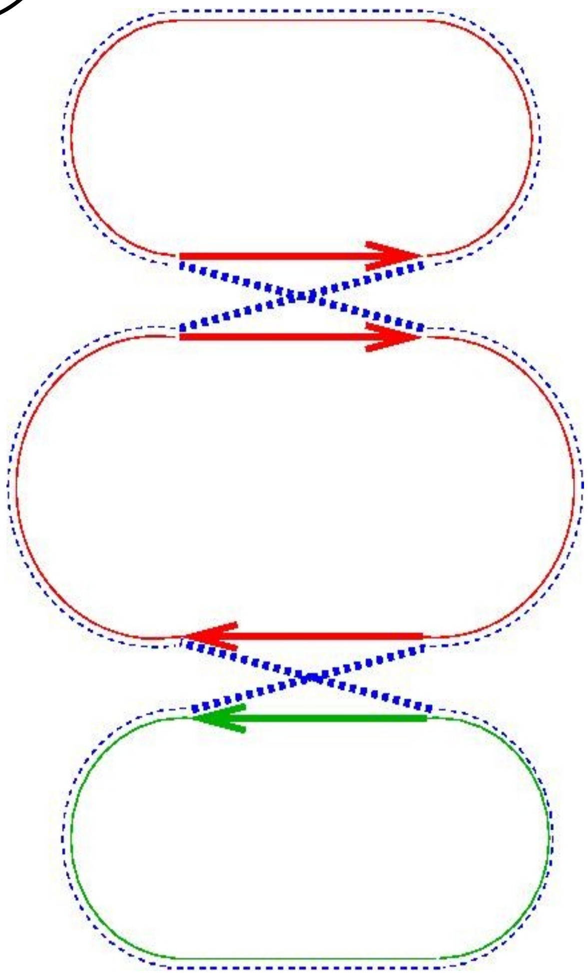

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Gaussian integral with powers

$$\int d^2 B_{jk} e^{-|B_{jk}|^2} (B_{jk})^m (B_{jk}^*)^n = \delta_{m,n} m!$$

Drawing orbits

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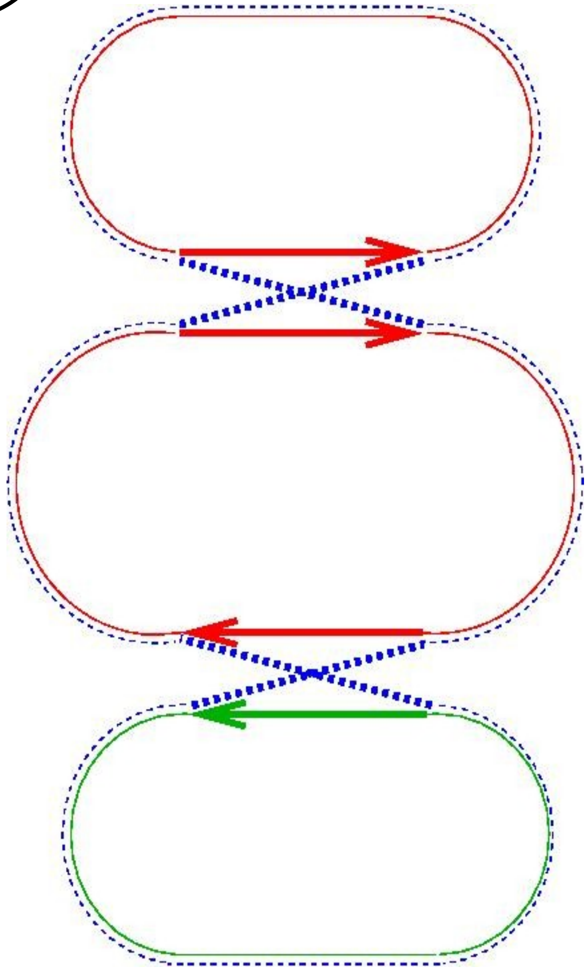

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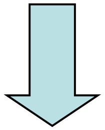

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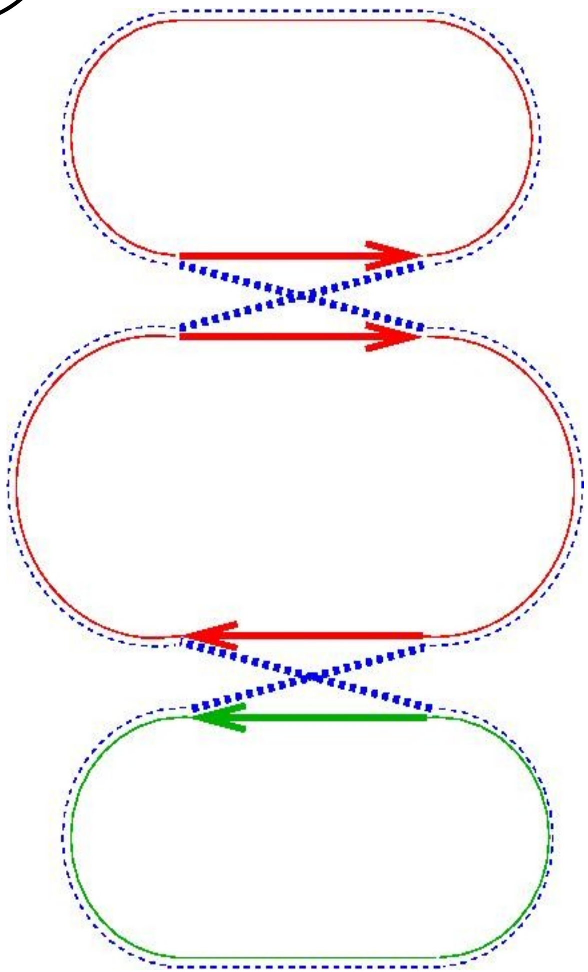

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also put in energy differences

Result

Result

Summation gives

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$$Z_1 = \int d[B] \exp \left[\mathrm{tr} X' \left(\frac{1}{2} + \sum_{l=1}^{\infty} (BB^\dagger)^l \right) + \mathrm{tr} X'' \left(\frac{1}{2} + \sum_{l=1}^{\infty} (B^\dagger B)^l \right) \right]$$

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Agreement with sigma model, RMT

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Difference to

ballistic sigma model:

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Muzykantskii & Khmel'nitskii, JETP Lett. (1995)
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- perturbative
- no problems due to regularisation

Outlook

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possible extension: **localization** e.g. in long wires

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- semiclassical contributions changed (diffusion)

Outlook

possible extension: **localization** e.g. in long wires

- semiclassical contributions changed (diffusion)
- expect one-dimension sigma model

Conclusions

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Semiclassics

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Semiclassics

⇩

Sigma model

Conclusions

Semiclassics

⇓

Sigma model

⇓

Universality

Conclusions

Semiclassics

⇩

Sigma model

⇩

Universality

- Same relation between non-osc. & osc. contributions

Conclusions

Semiclassics

⇩

Sigma model

⇩

Universality

- Same relation between non-osc. & osc. contributions

- encounters = perturbation series

Conclusions

Semiclassics

⇓

Sigma model

⇓

Universality

- Same relation between non-osc. & osc. contributions
- encounters = perturbation series
- count link connections using matrix integral