Parametric Spectral Correlation with Spin 1/2

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§ Periodic Orbit Theory

Let us consider the energy level statistics of a bounded quantum system with f degrees of freedom.

Each phase space point is specified by a vector $\mathbf{x} = (\mathbf{q}, \mathbf{p})$, where f dimensional vectors $\mathbf{q}$ and $\mathbf{p}$ give the position and momentum, respectively.

It is assumed that the corresponding classical dynamics is chaotic.

The system has a spin with a fixed quantum number S. The strength of the interaction between the spin and effective field is characterized by a parameter η.

The spin state is described by a spinor with $2S+1$ elements and the spin evolution operator $\hat{\Delta}$ is represented by a $(2S+1)\times(2S+1)$ matrix.

We denote such a representation matrix evaluated along the periodic orbit γ by $\Delta_\gamma(\eta)$.

Let us denote by E the energy of the system. Then, in the semiclassical limit $\hbar \to 0$, the energy level density $\rho(E;\eta)$ can be written in a decomposed form

$$\rho(E;\eta) \sim \rho_{\mathsf{av}}(E) + \rho_{\mathsf{osc}}(E;\eta).$$

Here $\rho_{\mathsf{av}}(E)$ is the local average of the level density, while $\rho_{\mathsf{osc}}(E;\eta)$ gives the fluctuation (oscillation) around the local average.

In the leading order of the semiclassical approximation, the fluctuation part of the level density is written as

$$\rho_{\mathsf{osc}}(E;\eta) = \frac{1}{\pi\hbar}\mathrm{Re}\sum_{\gamma}(\mathrm{tr}\Delta_\gamma(\eta))A_\gamma \mathrm{e}^{iS_\gamma(E)/\hbar}.$$

S_γ: the classical action for the orbital motion.

A_γ: the stability amplitude (including the Maslov phase).

Let us consider the spectral form factor

$$K(\tau;\eta,\eta') \;\sim\; \left\langle \int \mathrm{d}\epsilon\ \mathrm{e}^{i\epsilon\tau T_H/\hbar} \right.$$
$$\left. \times\quad \frac{\rho_{\mathsf{osc}}\left(E+\frac{\epsilon}{2};\eta\right)\rho_{\mathsf{osc}}\left(E-\frac{\epsilon}{2};\eta'\right)}{\rho_{\mathsf{av}}(E)} \right\rangle .$$

Here the angular brackets mean averages over windows of the center energy E and the time variable τ.

The scaled time τ is measured in units of the Heisenberg time

$$T_H = 2\pi\hbar\rho_{\mathsf{av}}(E).$$

It follows that the form factor is expressed as a double sum over periodic orbits

$$\begin{aligned} & K(\tau;\eta,\eta') \\ \sim\ & \frac{1}{T_H^2}\left\langle \sum_{\gamma,\gamma'} (\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_{\gamma'}(\eta'))^* A_\gamma A_{\gamma'}^* \right. \\ \times\ & \left. \mathrm{e}^{i(S_\gamma - S_{\gamma'})/\hbar}\delta\left(\tau - \frac{T_\gamma + T_{\gamma'}}{2T_H}\right)\right\rangle, \end{aligned}$$

where an asterisk stands for a complex conjugate.

The periods of the periodic orbit γ and its partner γ' are denoted by T_γ and $T_{\gamma'}$, respectively.

The time evolution of the spin is described by a $(2S+1)\times(2S+1)$ matrix $\Delta(t)$. Note that $\Delta(t)$ can be expressed as

$$\exp\left(i\phi(t)S_z/\hbar\right)\exp\left(i\theta(t)S_x/\hbar\right)\exp\left(i\psi(t)S_z/\hbar\right),$$

where S_x and S_z are $(2S+1)\times(2S+1)$ matrices representing the x and z components of the spin operator $\widehat{S}$.

Thus three Euler angles ψ, θ and ϕ describe the spin evolution.

In principle, the spin evolution matrix $\Delta_\gamma(\eta)$ can be calculated from a deterministic equation of motion.

However, we here simply assume that the spin evolution parameters undergo Brownian motion on the surface of a sphere.

Then the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \eta^2 D \mathcal{L}_{\mathrm{SP}} P$$

holds for the p.d.f.(probability distribution function) $P(\psi,\theta,\phi)$ with the measure

$$\sin\theta \mathrm{d}\psi \mathrm{d}\theta \mathrm{d}\phi.$$

Here D is the diffusion constant and

$$\begin{aligned}\mathcal{L}_{\mathrm{SP}} &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} \\ &+ \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\psi^2}+\frac{\partial^2}{\partial\phi^2}-2\cos\theta\frac{\partial^2}{\partial\psi\partial\phi}\right)\end{aligned}$$

is the Laplace-Beltrami operator on the sphere.

The Green function solution of the Fokker-Planck equation is known to be

$$\begin{aligned} & g(\psi, \theta, \phi; t | \psi', \theta', \phi') \\ = & \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \sum_{n=-j}^{j} \frac{2j+1}{32\pi^2} \\ \times & D^j_{m,n}(\psi, \theta, \phi) \left\{ D^j_{m,n}(\psi', \theta', \phi') \right\}^* \mathrm{e}^{-j(j+1)\eta^2 D t}, \end{aligned}$$

where $D^j_{m,n}$ is Wigner's D function.

Here j is an integer or a half odd integer ($j = 0, 1/2, 1, 3/2, \cdots$ and $m, n = -j, -j+1, \cdots, j$).

Replacing the factor

$$(\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_{\gamma'}(\eta'))^*$$

by the average

$$\langle\langle\ (\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_{\gamma'}(\eta'))^*\ \rangle\rangle$$

over the Brownian motion, we can write the form factor as

$$\begin{aligned} & K(\tau;\eta,\eta') \\ \sim\ & \frac{1}{T_H^2}\left\langle \sum_{\gamma,\gamma'} \langle\langle\ (\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_{\gamma'}(\eta'))^*\ \rangle\rangle \right. \\ \times\ & \left. A_\gamma A_{\gamma'}^* \mathrm{e}^{i(S_\gamma - S_{\gamma'})/\hbar} \delta\left(\tau - \frac{T_\gamma + T_{\gamma'}}{2T_H}\right)\right\rangle . \end{aligned}$$

We shall evaluate the τ expansion of this semi-classical form factor, focusing on the systems with spin $S = 1/2$.

§ Diagonal Approximation

The leading term in the τ expansion can be evaluated by using Berry's diagonal approximation(Proc. R. Soc. London A400 (1985) 229).

In Berry's diagonal approximation, one only takes account of the periodic orbit pairs

$$(\gamma, \gamma)$$

and

$$(\gamma, \bar{\gamma}),$$

where a bar denotes time reversal.

Let us first consider the contributions from the pairs of identical periodic orbits (γ, γ).

The spin evolution matrix along γ with $S = 1/2$ is given by

$$\Delta_\gamma(\eta) = \exp\left(\phi\frac{i}{2}\sigma_z\right)\exp\left(\theta\frac{i}{2}\sigma_x\right)\exp\left(\psi\frac{i}{2}\sigma_z\right),$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices.

The average of the factor $(\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_\gamma(0))$ can be written as

$$\begin{aligned} & \langle\langle\ (\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_\gamma(0))\ \rangle\rangle \\ = & \int \mathrm{d}\omega\mathrm{d}\omega'(\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_\gamma(0)) \\ \times & \ g(\psi,\theta,\phi;T|\psi',\theta',\phi')p_0(\psi',\theta',\phi'), \end{aligned}$$

where p_0 is the p.d.f. of the Euler angles at $\eta = 0$ and $T = T_\gamma$ is the period of γ.

We employ the uniform "initial distribution"

$$p_0(\psi, \theta, \phi) = \frac{1}{32\pi^2},$$

which gives the transition within the GSE (Gaussian Symplectic Ensemble) universality class.

Then we find

$$\langle\langle\ (\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_\gamma(0))\ \rangle\rangle = \mathrm{e}^{-(3/4)aT}$$

with

$$a = \eta^2 D.$$

Here η is scaled so that aT remains finite in the limit $\hbar \to 0$.

Using Hannay and Ozorio de Almeida (HOdA)'s sum rule

$$\frac{1}{T_H^2}\left\langle \sum_\gamma |A_\gamma|^2\,\delta\left(\tau - \frac{T_\gamma}{T_H}\right)\right\rangle = \tau,$$

resulting from the ergodicity of the system, we obtain the contribution to the form factor as

$$K_{(\gamma,\gamma)}(\tau;\eta,0) = \tau \mathrm{e}^{-(3/4)aT}.$$

Moreover, noting the symmetry

$$\mathrm{tr}\Delta_{\bar{\gamma}}(\eta) = \mathrm{tr}\Delta_\gamma(\eta),$$

we find the total contribution from the diagonal approximation

$$\begin{aligned} K_{\mathrm{diag}}(\tau) &= K_{(\gamma,\gamma)}(\tau;\eta,0) + K_{(\gamma,\bar{\gamma})}(\tau;\eta,0) \\ &= 2\tau \mathrm{e}^{-(3/4)aT}. \end{aligned}$$

§ **Off-diagonal Contributions**

In the leading off-diagonal terms, we suppose that γ' is almost identical to γ or $\bar{\gamma}$ on the loops but differently connected in the encounters.

The simplest example of such a pair (γ, γ') has two loops (L_1 and L_2) and one encounter (E_1) (Sieber and Richter, Physica Scripta T90 (2001) 128).

As before, we need to evaluate the average

$$\langle\langle\ (\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_{\gamma'}(0))\ \rangle\rangle.$$

for this Sieber-Richter pair.

We symbolically write the periodic orbits of the Sieber-Richter pair as

$$\gamma = \bar{E}_1 L_2 E_1 L_1, \quad \gamma' = \bar{E}'_1 \bar{L}'_2 E'_1 L'_1,$$

so that the spin evolution matrices are

$$\begin{aligned} \Delta_\gamma &= (\Delta_{E_1})^{-1} \Delta_{L_2} \Delta_{E_1} \Delta_{L_1}, \\ \Delta_{\gamma'} &= (\Delta_{E'_1})^{-1} (\Delta_{L'_2})^{-1} \Delta_{E'_1} \Delta_{L'_1}. \end{aligned}$$

Using the above formulas, we evaluate the average as (t_1 is the duration of E_1)

$$\begin{aligned} & \langle\langle\ (\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_{\gamma'}(0))\ \rangle\rangle \\ = \ & \frac{1}{4} \mathrm{e}^{-(3/4)aT} \left(\mathrm{e}^{(3/2)at_1} - 3\mathrm{e}^{-(1/2)at_1} \right). \end{aligned}$$

Using the strategy of Müller et al. (Phys. Rev. Lett. **93** (2004) 014103), we find the contribution from the Sieber-Richter pair as

$$K_{\mathrm{SR}}(\tau) = 2\tau^2 \mathrm{e}^{-(3/4)aT} \left(1 + \frac{3}{4} aT \right).$$

The next term $K_{\mathrm{3rd}}(\tau)$ of the order τ^3 was also calculated in the Reference (J. Phys. A: Math. Theor. 40 (2007) 12055) by using more complicated diagrams.

As a result, the semiclassical form factor up to the order τ^3 is evaluated as

$$\begin{aligned} K_{\mathrm{SC}}(\tau) &= K_{\mathrm{diag}}(\tau) + K_{\mathrm{SR}}(\tau) + K_{\mathrm{3rd}}(\tau) \\ &= 2\tau \mathrm{e}^{-(3/4)aT} \left[1 + \left(1 + \frac{3}{4}aT \right) \tau \right. \\ &+ \left. \left\{ 1 + \frac{3}{4}aT + \frac{15}{32}(aT)^2 \right\} \tau^2 \right]. \end{aligned}$$

For the transition within the GSE universality class, the prediction of parametric random matrices can be written as (Simons, Lee and Altshuler, Phys. Rev. B48 (1993) 11450)

$$\begin{aligned} K_{\mathsf{RM}}(\tau) &= 2\tau \mathrm{e}^{-2\lambda} \{1 + (1+2\lambda)\tau \\ &+ \left(1 + 2\lambda + \frac{10}{3}\lambda^2\right)\tau^2 + \cdots \right\} \end{aligned}$$

with a transition parameter λ.

This is in agreement with the semiclassical formula up to the third order with an identification $\lambda = (3/8)aT$.

§ The GOE to GSE Transition

If the spin evolution operator is represented by an identity matrix, the system is effectively spinless.

A spinless system is described by the GOE(Gaussian Orthogonal Ensemble) universality class of random matrices.

Therefore, the crossover from the GOE class to the GSE class can be treated by introducing

$$p_0(\psi, \theta, \phi) = \delta(\psi)\delta(\cos\theta - 1)\delta(\phi)$$

as the "initial distribution".

Let us calculate the form factor $K(\tau, \eta, \eta)$, where η' is equated with η.

For the diagonal terms, the necessary averages over the Brownian motion can be evaluated as

$$\begin{aligned} \langle\langle\ (\mathrm{tr}\Delta_\gamma(\eta))^2\ \rangle\rangle &= \langle\langle\ (\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_{\bar\gamma}(\eta))\ \rangle\rangle \\ &= 1 + 3\mathrm{e}^{-2aT}. \end{aligned}$$

On the other hand, for the Sieber-Richter term, we find (T_1 is the duration of L_1)

$$\begin{aligned} & \langle\langle\ (\mathrm{tr}\Delta_\gamma(\eta))(\mathrm{tr}\Delta_{\gamma'}(\eta))\ \rangle\rangle \\ = & -\frac{1}{2} + \frac{3}{2}\mathrm{e}^{-2aT+4at_1} \\ + & \frac{3}{2}\mathrm{e}^{-2aT+2aT_1+4at_1} + \frac{3}{2}\mathrm{e}^{-2aT_1}. \end{aligned}$$

Then we obtain the semiclassical form factor up to the second order as

$$\begin{aligned} K_{\mathrm{SC}}(\tau) &= 2\tau(1+3\mathrm{e}^{-2aT}) \\ &+ 2\tau^2\left\{1+(6aT-9)\mathrm{e}^{-2aT}\right\}. \end{aligned}$$

It is expected to give a universal formula describing the GOE to GSE transition.