Quantum symbolic dynamics

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Quantum chaos: routes to RMT and beyond
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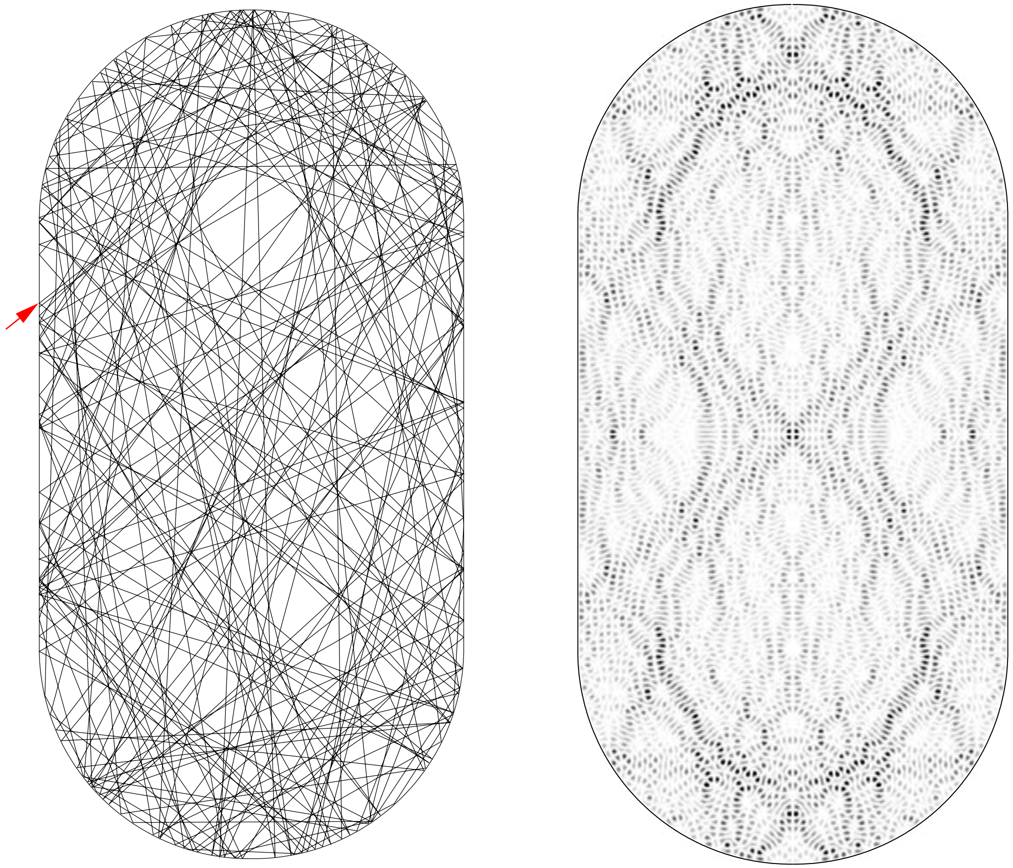
What do we know about chaotic eigenstates?

- Hamiltonian $H(q,p)$, such that the dynamics on Σ_E is chaotic. $H_\hbar = \mathrm{Op}_\hbar(H)$ has discrete spectrum $(E_{\hbar,n}, \psi_{\hbar,n})$ near the energy E.

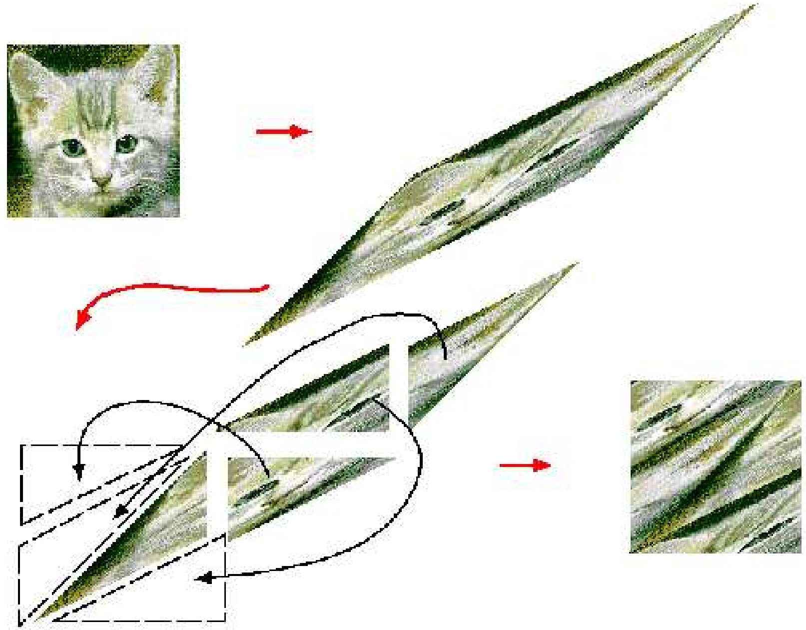
- Laplace operator on a "chaotic cavity", or on a surface of negative curvature.

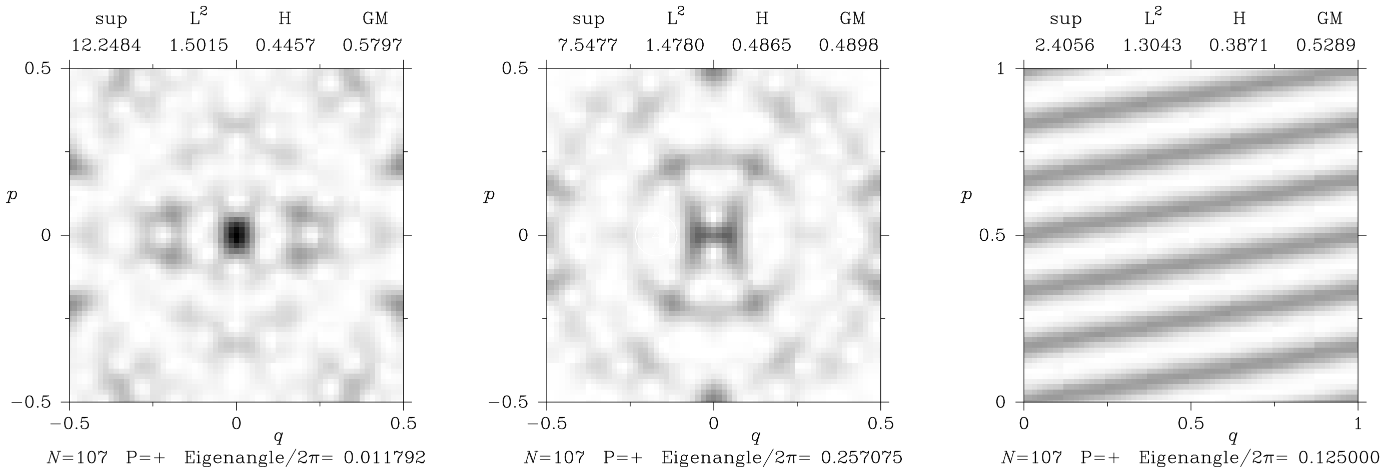


What do the eigenstates $\psi_\hbar$ look like in the *semiclassical limit* $\hbar \ll 1$.?

Chaotic quantum maps

• chaotic map Φ on a compact phase space $\rightsquigarrow$ propagators $U_N(\Phi)$, $N \sim \hbar^{-1}$ (advantage: easy numerics, some models are "partially solvable").





Husimi densities of some eigenstates.

Phase space localization

Interesting to study the localization of $\psi_\hbar$ in both position and momentum: **phase space description**.

Ex: for any bounded test function (observable) $f(q,p)$, study the matrix elements

$$f(\psi_\hbar) = \langle \psi_\hbar, \mathrm{Op}_\hbar(f)\, \psi_\hbar \rangle = \int dq dp\, f(q,p)\, \rho_{\psi_\hbar}(q,p)$$

Depending on the quantization, the function $\rho_{\psi_\hbar}$ can be the Wigner function, the Husimi function.

Def: from any sequence $(\psi_\hbar)_{\hbar\to 0}$, one can always extract a subsequence $(\psi_{\hbar'})$ such that

$$\text{for any } f, \quad \lim_{\hbar'\to 0} f(\psi_{\hbar'}) = \mu(f)$$

μ is a measure on phase space, called the **semiclassical measure** of the sequence $(\psi_{\hbar'})$.

μ takes the macroscopic features of $\rho_{\psi_\hbar}$ into account. Fine details (e.g. oscillations, correlations, nodal lines) have disappeared.

Quantum-classical correspondence

For the eigenstates of $H_\hbar$, μ is supported on Σ_E.

Let us use the flow Φ^t generated by H, and call $U^t = e^{-itH_\hbar/\hbar}$ the quantum propagator.

Egorov's theorem: for any observable f,

$$U^{-t}\,\mathrm{Op}_\hbar(f)\,U^t = \mathrm{Op}_\hbar(f \circ \Phi^t) + \mathcal{O}(\hbar\, e^{\Lambda t})$$

Idem for a quantum map $U = U_N(\Phi)$.

Breaks down at the Ehrenfest time $T_E = |\log \hbar|/\Lambda$ (cf. R.Whitney's talk).

$\rightsquigarrow$ the semiclassical measure μ is thus **invariant through the classical dynamics**.

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If Φ is *ergodic w.r.to the Liouville measure*, one can show (again, using Egorov) that **almost all** eigenstates $\psi_{\hbar,n}$ become equidistributed when $\hbar \to \infty$:

$$\forall f, \qquad N^{-1} \sum_{n=1}^{N} |f(\psi_{\hbar,n}) - \int f\, d\mu_L|^2 \xrightarrow{h \to 0} 0.$$

Quantum ergodicity [Shnirelman'74, Zelditch'87, Colin de Verdière'85]

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Quantum ergodicity [Shnirelman'74, Zelditch'87, Colin de Verdière'85]

$\rightarrow$ do ALL eigenstates become equidistributed [Rudnick-Sarnak'93]? Or are there exceptional sequences of eigenstates?

Some counter-examples

No exceptional sequences for *arithmetic* eigenstates of (2D) cat maps [Rudnick-Sarnak'00] and for *arithmetic* surfaces [Lindenstrauss'06].

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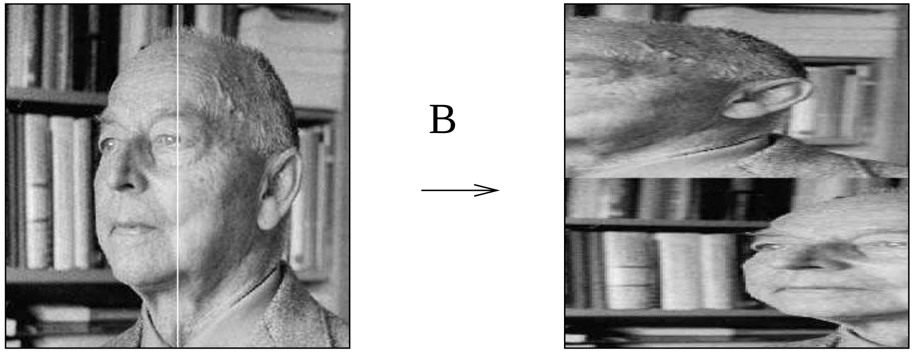
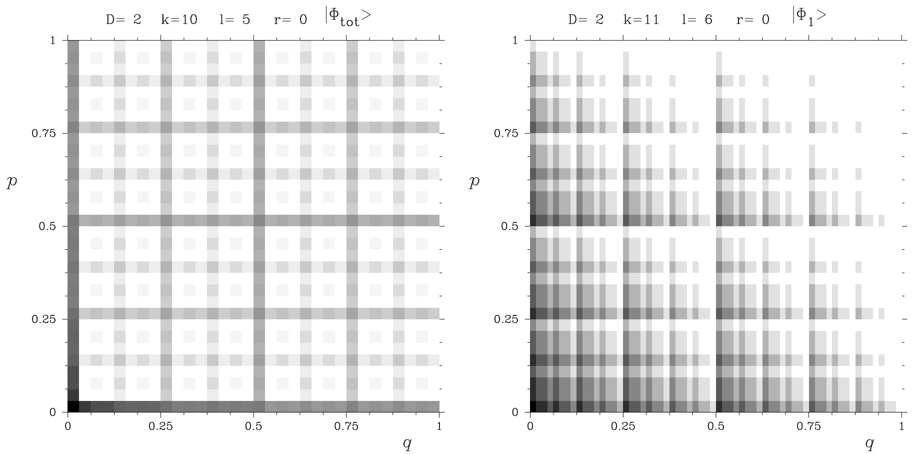
$\exists$ explicit exceptional semiclassical measures for the quantum cat map [Faure-N-DeBièvre] and Walsh-quantized baker's map [Anantharaman-N'06] (cf. Kelmer's talk).





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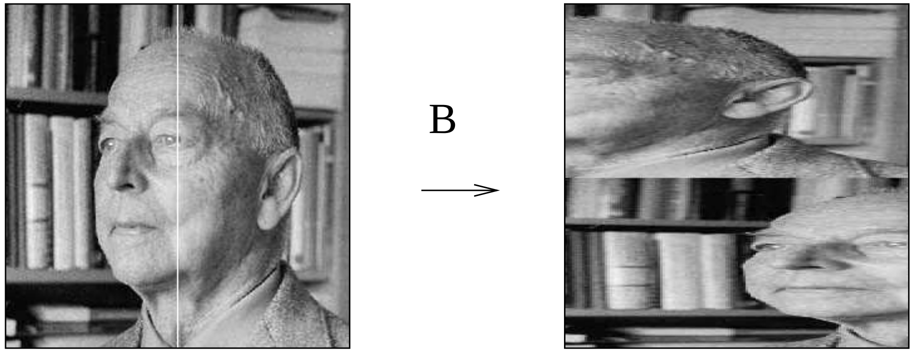
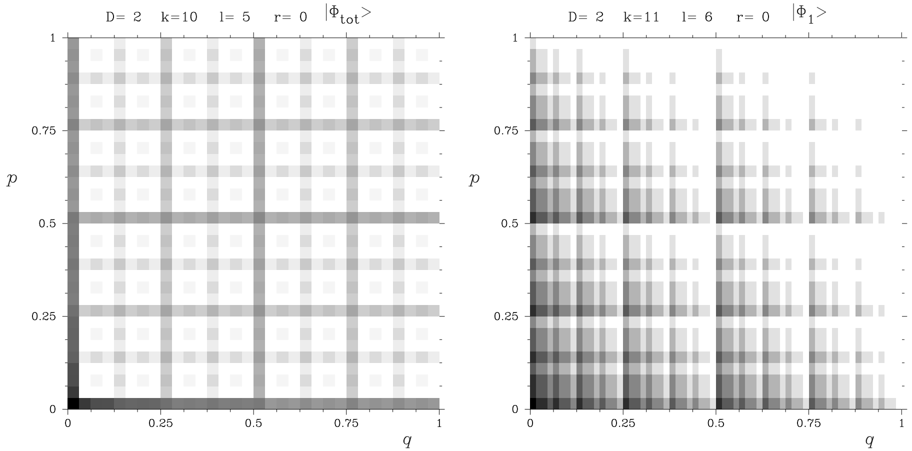


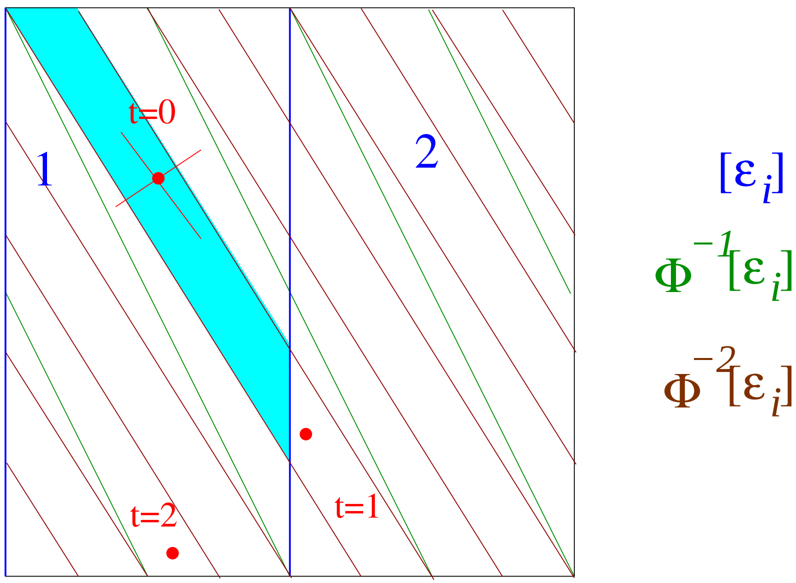


$\rightarrow$ in general, can ANY invariant measures occur as a semiclassical measure?

In particular, can one have strong scars $\mu_{sc} = \delta_{PO}$?

Symbolic dynamics of the classical flow

To classify Φ-invariant measures, one may use a phase space partition; each trajectory will be represented by a symbolic sequence $\cdots \epsilon_{-1}\epsilon_0\epsilon_1 \cdots\cdots$ denoting its "history".



At each time n, the *rectangle* $[\epsilon_0 \cdots \epsilon_n]$ contains all points sharing the same history between times 0 and n (ex: [121]).

Let μ be an invariant proba. measure. The time-n entropy

$$H_n(\mu) = -\sum_{\epsilon_0,\ldots,\epsilon_n} \mu([\epsilon_0 \cdots \epsilon_n]) \, \log \mu([\epsilon_0 \cdots \epsilon_n])$$

measures the distribution of the weights $\mu([\epsilon_0 \cdots \epsilon_n])$.

KS entropy of semiclassical measures

The Kolmogorov-Sinai entropy $H_{KS}(\mu) = \lim_n n^{-1} H_n(\mu)$ represents the "information complexity" of μ w.r.to the flow.

- Related to localization: $H_{KS}(\delta_{PO}) = 0$, $H_{KS}(\mu_L) = \int \log J^u \, d\mu_L$.
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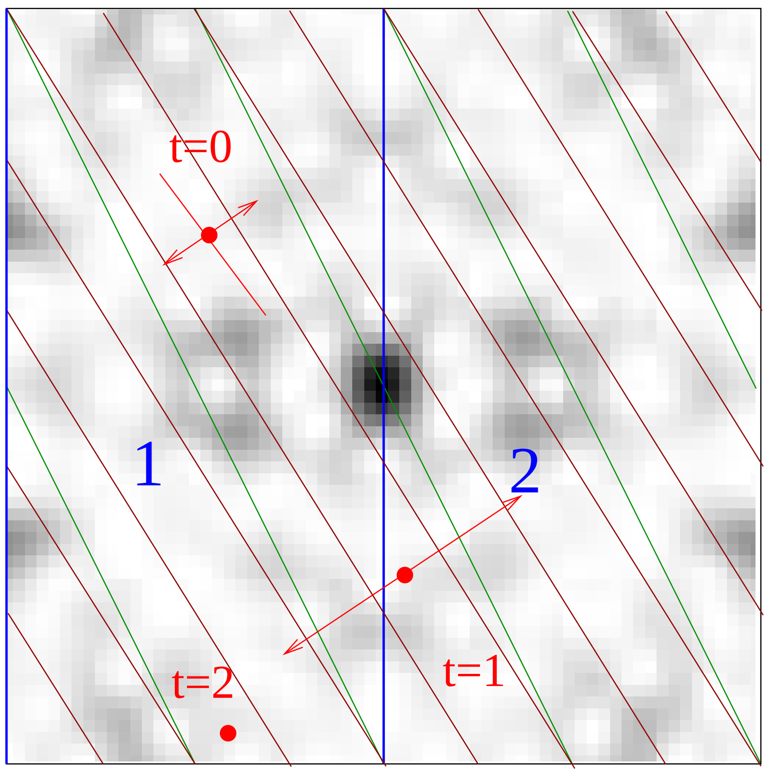


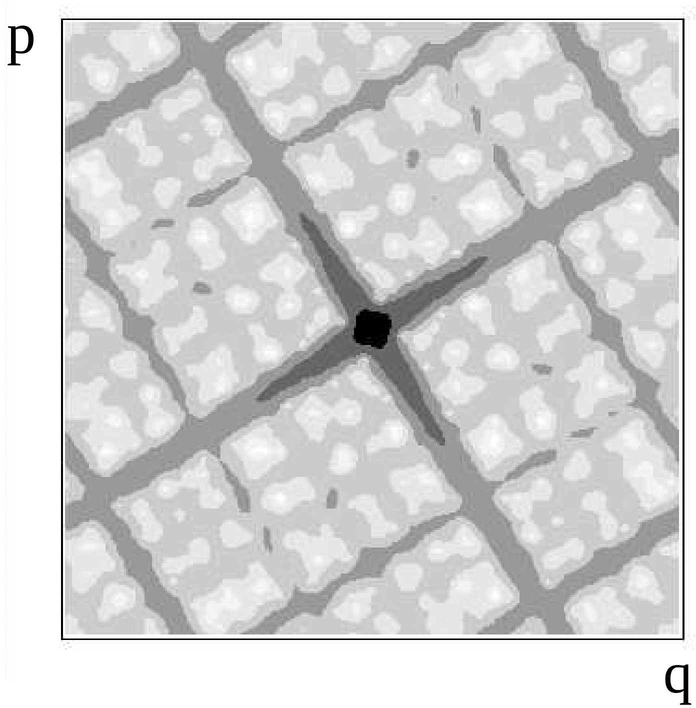
What can be the entropy of a semiclassical measure for an Anosov system?

Semiclassical measures are at least "half-delocalized"

<u>Theorem</u> [Anantharaman-Koch-N'07]: For any quantized Anosov system, any semiclassical measure μ satisfies

$$H_{KS}(\mu) \geq \int \log J^u \, d\mu - \frac{1}{2}\Lambda_{\max}(d-1)$$

$\rightsquigarrow$ "full scars" are forbidden. Some of the exceptional measures saturate this lower bound.



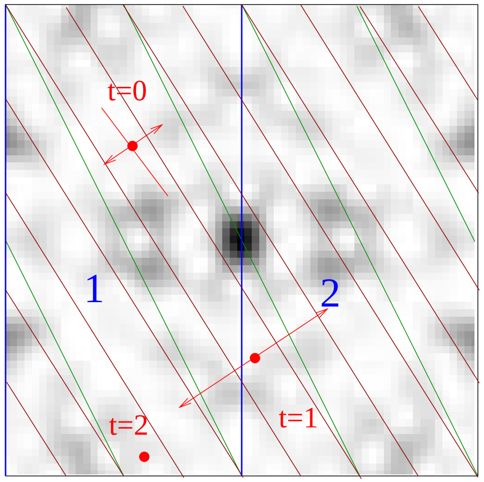
Quantum partition of unity

Using quasi-projectors $P_j = \mathrm{Op}_\hbar(\chi_j)$ on the components of the partition, we construct a **quantum partition of unity** $Id = \sum_{j=1}^J P_j$.

Egorov thm $\Rightarrow$ for $n < T_E$ the operator

$$P_{\epsilon_0\cdots\epsilon_n} = U^{-n}\tilde{P}_{\epsilon_0\cdots\epsilon_n} \stackrel{\text{def}}{=} U^{-n}P_{\epsilon_n}U\cdots P_{\epsilon_1}UP_{\epsilon_0}$$

is a quasi-projector on the rectangle $[\epsilon_0\cdots\epsilon_n]$.



Can we get some information on the distribution of the weights $\|P_{\epsilon_0\cdots\epsilon_n}\psi_\hbar\|^2 \xrightarrow{h\to 0} \mu([\epsilon_0\cdots\epsilon_n])$?

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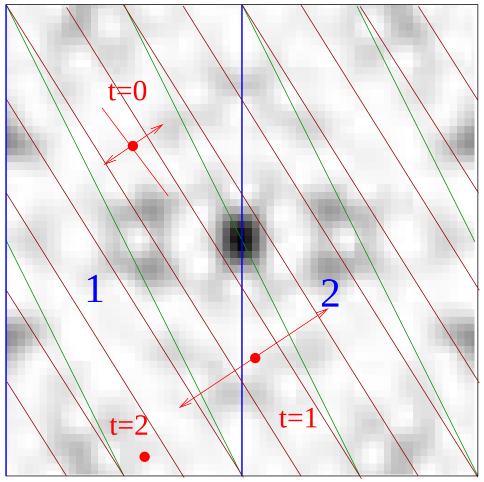

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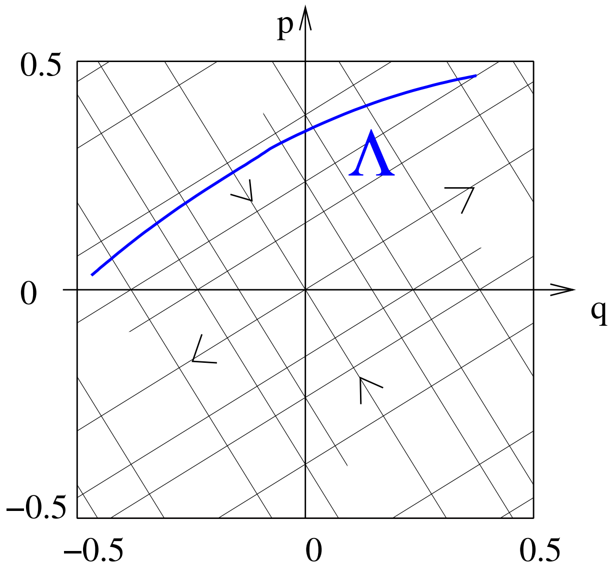
$\|P_{\epsilon_0 \cdots \epsilon_n} \psi_\hbar\|^2 \xrightarrow{h \to 0} \mu([\epsilon_0 \cdots \epsilon_n])$?
YES, provided we consider times $n > T_E$ (for which the quasi-projector interpretation breaks down).

Evolution of "adapted elementary states"

To estimate $P_{\epsilon_0 \cdots \epsilon_n} \psi_\hbar$, we decompose $\psi_\hbar$ in a well-chosen family of states ψ_Λ, and compute each $P_{\epsilon_0 \cdots \epsilon_n} \psi_\Lambda$ separately.

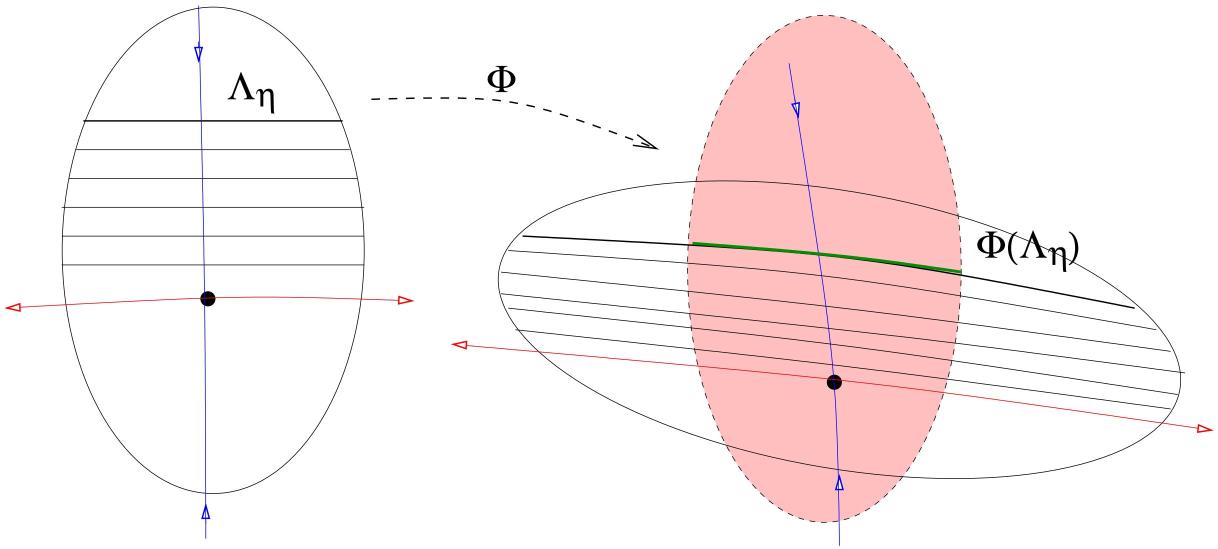
The Anosov dynamics is anisotropic (stable/unstable foliations).



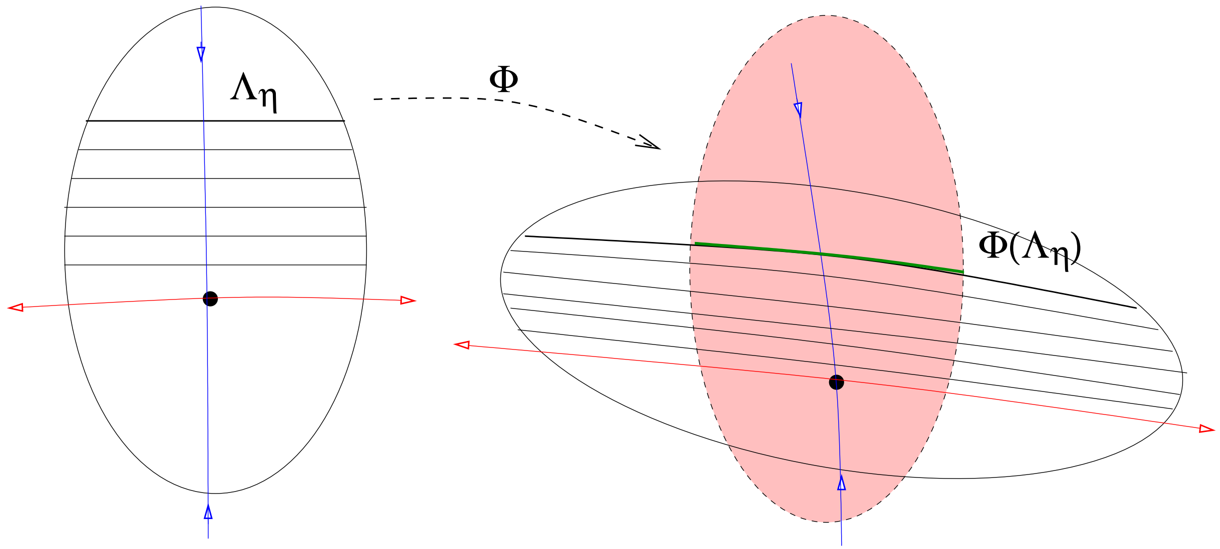
$\Rightarrow$ use states adapted to these foliations.

We consider Lagrangian states associated with Lagrangian manifolds "close to" the unstable foliation:

$$\psi_\Lambda(q) = a(q)\, e^{iS_\Lambda(q)/\hbar} \quad \text{is localized on } \Lambda = \{(q, p = \nabla S_\Lambda(q))\}$$



Through the sequence of stretching-and-cutting, the state $\tilde{P}_{\epsilon_0\cdots\epsilon_n}\psi_\Lambda$ remains a nice Lagrangian state up to large times ($n \approx C\,T_E$ for any $C > 1$).



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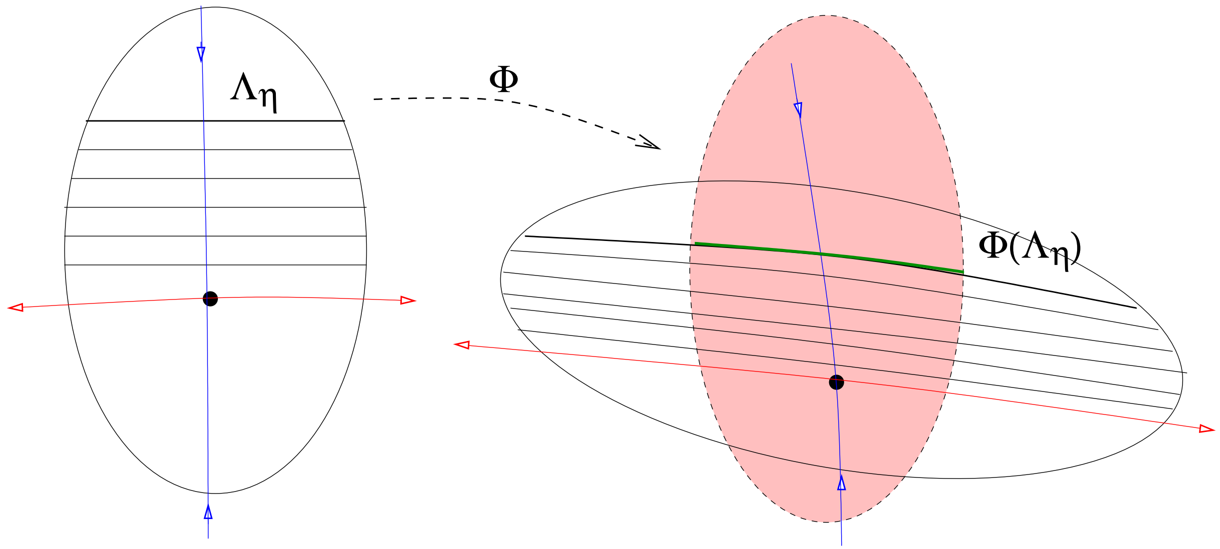
• Summing over all $[\epsilon_0 \cdots \epsilon_n]$ we recover $U^n \psi_\Lambda$: no breakdown of the semiclassical evolution at T_E [Heller-Tomsovic'91].



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• The amplitude of $\tilde{P}_{\epsilon_0\cdots\epsilon_n}\psi_\Lambda$ is governed by the unstable Jacobian along the path $\epsilon_0 \cdots \epsilon_t$:

$$\|\tilde{P}_{\epsilon_0\cdots\epsilon_n}\psi_\Lambda\| \sim J_u^n(\epsilon_0\cdots\epsilon_n)^{-1/2} \sim e^{-\Lambda n/2}$$

From the decomposition $\psi_\hbar = \sum_{\eta=1}^{1/h} c_\eta\, \psi_{\Lambda_\eta}$, one obtains the bound [Anantharaman'06]

$$\boxed{\|P_{\epsilon_0\cdots\epsilon_n}\| \leq h^{-1/2}\, J_u^n(\epsilon_0\cdots\epsilon_n)^{-1/2} \sim h^{-1/2}\, e^{-\Lambda n/2}}.$$

This "hyperbolic estimate" is **nontrivial for times** $t > T_E$ (no more a quasi-projector).

What to do with this estimate?

[Anantharaman-N'06'07]: rewrite the operator at time $2n$ as

$$U^n P_{\epsilon_0\cdots\epsilon_{2n}} = P_{\epsilon_{n+1}\cdots\epsilon_{2n}} U^n P_{\epsilon_0\cdots\epsilon_n}$$

This can be seen as a "block matrix element" of the unitary propagator U^n, expressed in the block-basis $\{P_{\epsilon_0\cdots\epsilon_n}\}$.

Setting $n = T_E$, the hyperbolic estimate at time $2n$ states that these "block matrix elements" are all $\leq \hbar^{1/2}$.

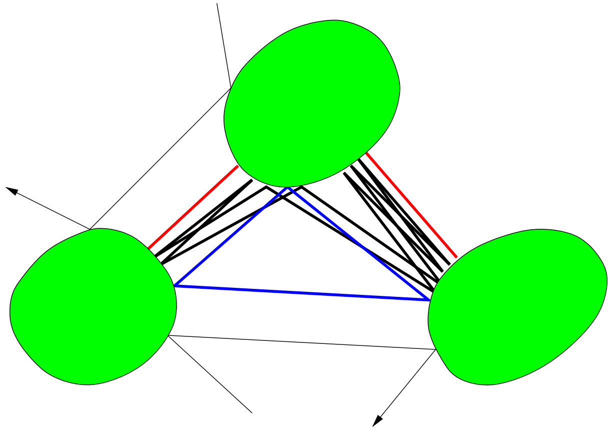
An **entropic uncertainty principle** then implies that the entropy constructed from the weights $\|P_{\epsilon_0\cdots\epsilon_n}\psi_\hbar\|^2$ satisfies

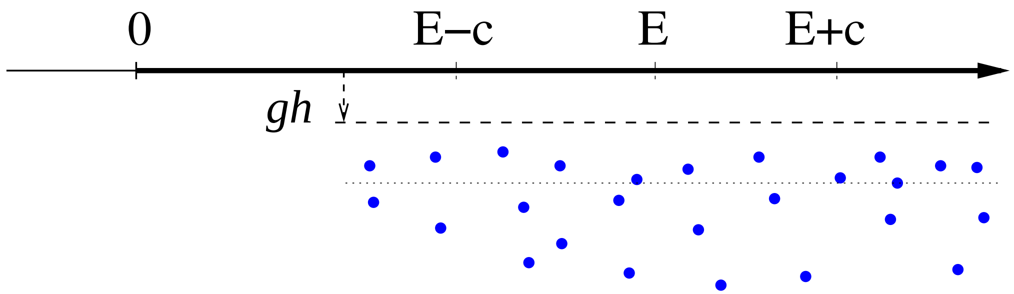
$$H_n(\psi_\hbar) \geq |\log \hbar^{1/2}| = \frac{n\,\Lambda}{2}\,.$$

From this bound at $n = T_E$, one uses subadditivity and Egorov to get a similar bound at finite time n, and then the bound for $H_{KS}(\mu)$. □

Where else to use this hyperbolic estimate?

• semiclassical resonance spectra of chaotic scattering systems.





Discrete model: open quantum baker

⇝ subunitary propagator $B_N = U_N \circ \Pi_N$.

Gap in the resonance spectrum

Use a quantum partition outside the hole: $\Pi_N = \sum_j P_j$.

$$
\begin{aligned}
(B_N)^n &= \sum_{\epsilon_0 \cdots \epsilon_{n-1}} \tilde{P}_{\epsilon_0 \cdots \epsilon_{n-1}} \\
\Longrightarrow \|(B_N)^n\| &\leq \sum_{\epsilon_0 \cdots \epsilon_{n-1}} \|\tilde{P}_{\epsilon_0 \cdots \epsilon_{n-1}}\| \\
&\leq \sum_{\epsilon_0 \cdots \epsilon_{n-1}} h^{-1/2}\, J_u^n(\epsilon_0 \cdots \epsilon_{n-1})^{-1/2}
\end{aligned}
$$

For $n \gg T_E$, the RHS is approximately given by the topological pressure $\mathcal{P}(-\log J_u/2)$ associated with the classical trapped set:

$$
\|(B_N)^n\| \leq \exp\big(n\, \mathcal{P}(-\log J_u/2)\big)
$$

If $\mathcal{P}(-\log J_u/2) < 0$ ("thin trapped set"), this gives an upper bound on the quantum lifetimes [Ikawa'88, Gaspard-Rice'89, N-Zworski'07].

Perspectives

To obtain nontrivial information on eigenstates, it was crucial to analyze the dynamics **beyond the Ehrenfest time**.

The partition allows to control the evolution of Lagrangian states (also wavepackets) [Heller-Tomsovic'91,Schubert'08]

The decomposition into $\sum P_{\epsilon_0 \cdots \epsilon_n}$ could also be useful to:

- analyze the phase space structure of resonant states [Keating-Novaes-Prado-Sieber'06, N-Rubin'06]
- expand the validity of the Gutzwiller trace formula to times $n \gg T_E$ [Faure'06]
- show that (some) resonances are close to the zeros of the Gutzwiller-Voros Zeta function
- analyze transport through chaotic cavities