Periodic orbit encounters: a mechanism for trajectory correlations

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Wigner time delay

\[ \tau_W(E) = - \frac{i\hbar}{M} \text{Tr} \left[ S^\dagger(E) \frac{d}{dE} S(E) \right] \]

with scattering matrix \( S(E) \) and \( M \) open scattering channels.

Relation to the ‘density of states’ (Friedel (1952))

\[ \tau_W(E) = \frac{2\pi\hbar}{M} d(E) \approx \frac{2\pi\hbar}{M} (\bar{d}(E) + d^{\text{osc}}(E)) \]

The mean time delay is

\[ \bar{\tau}_W(E) \approx \frac{1}{\mu} \]

where \( \mu \) is the classical escape rate.
The two semiclassical formulas

The semiclassical formula for the elements of the scattering matrix is

\[ S_{ba}(E) \approx \frac{1}{\sqrt{T_H}} \sum_{\alpha(a \to b)} A_\alpha e^{\frac{i}{\hbar} S_\alpha} e^{-\frac{i\pi}{2} \nu_\alpha} \]

where \( T_H = 2\pi \hbar d \) is the Heisenberg time.

We arrive at two different semiclassical formulae for the Wigner time delay, and we expect the following to hold

\[ \frac{1}{MT_H} \sum_{a,b} \sum_{\alpha,\alpha'} T_\alpha A_\alpha A^*_\alpha' e^{\frac{i}{\hbar} (S_\alpha - S_{\alpha'})} e^{-\frac{i\pi}{2} (\nu_\alpha - \nu_{\alpha'})} \]

\[ \approx \bar{\tau}_W + \frac{2}{M} \text{Re} \sum_{p,r} A_{p,r}(E) e^{\frac{i}{\hbar} r S_p(E)} e^{-\frac{i\pi}{2} r \mu_p} \]

We will start from the double sum over scattering trajectories and derive all terms in the periodic orbit formula.
The correlation function $C(\epsilon)$

Instead of working with the time delay, it is convenient to consider instead the following correlation function

$$C(\epsilon) = \sum_{a,b} S_{ba} \left( E + \frac{\epsilon \mu \hbar}{2} \right) S^*_{ba} \left( E - \frac{\epsilon \mu \hbar}{2} \right)$$

where $\mu$ is the classical escape rate. Using the unitarity of the scattering matrix, one obtains the Wigner time delay as

$$\tau_W = \left. \frac{-i}{\mu M} \frac{d}{d\epsilon} C(\epsilon) \right|_{\epsilon=0}$$

The semiclassical approximation for $C(\epsilon)$ is very similar to that of the Landauer-Büttiker conductance which is proportional to

$$\mathcal{T} = \sum_{b_{\text{out}}a_{\text{in}}} \langle S_{b_{\text{out}}a_{\text{in}}}(E) S^*_{b_{\text{out}}a_{\text{in}}}(E) \rangle$$
The diagonal approximation

A trajectory is paired only with itself (or its time-reverse). One uses a sum rule for open trajectories which is based on the ergodic exploration of the available phase space plus the finite escape probability

\[ \sum_{\alpha(a \rightarrow b)} |A_\alpha|^2 e^{i\epsilon \mu T_\alpha} \approx \int_0^\infty dT \ e^{-\mu T} \ e^{i\epsilon \mu T} \]

The sum over channels gives a factor $M^2$ for systems without TRS ($\kappa = 1$), and a factor of $M(M + 1)$ for systems with TRS ($\kappa = 2$), because one can pair a trajectory with its time-reverse if $a = b$

\[ C_{\text{diag}}(\epsilon) \approx \frac{(M + \kappa - 1)}{(1 - i\epsilon)} \]

This yields the correct mean time delay $\mu^{-1}$ for systems without TRS, but it is slightly wrong for systems with TRS (the numerator should be $M$)
Off-diagonal terms for $\tau_W$

The off-diagonal contributions come from trajectories with self-encounters and their partner orbits. Similar to conductance (Richter, M.S. (2002); Heusler et al. (2006); Müller et al. (2007)).

The number of $l$-encounters are collected in a vector $\nu$.

Diagrammatic rules:

For each link: \[ [M (1 - i\epsilon)]^{-1} \]

For each $l$-encounter: \[ -M (1 - il\epsilon) \]

With a sum rule for the number of structures $\tilde{N}(\nu)$ one arrives at

\[ \tilde{C}(\epsilon) \approx M [1 + i\epsilon + O(\epsilon^2)] \]
Periodic orbit encounters

For the periodic orbit contributions we consider trajectories that approach a periodic orbit $p$, follow it a number of times, and leave it again.

The Poincaré map has a simple form in the vicinity of $p$

$$s' = \Lambda_p^{-1} s, \quad u' = \Lambda_p u$$

where $\Lambda_p = \pm e^{\lambda_p T_p}$ is an eigenvalue of the stability matrix $M_p$. 
The trajectory pairs

Consider an orbit that has $k$ intersections in the Poincaré surface, $P_1, \ldots, P_k$, limited by the constant $c$. Its partner orbit has $r$ more intersections

$$P_1 = (u_1, s_1) \implies P'_1 \approx (\Lambda_p^{-r} u_1, s_1)$$

The action difference is

$$\Delta S = S_\alpha - S_{\alpha'} = su(1 - \Lambda_p^{-r}) - rS_p$$

One can define an encounter time for $\alpha$ which is given by

$$t_{\text{enc}}^p(s, u) = k T_p \approx \frac{1}{\lambda_p} \ln \frac{c^2}{us}$$
The semiclassical contribution

The semiclassical amplitudes are proportional to the $M_{12}$-element of the stability matrix. We can write $M_\alpha = M_f M_p^k M_i$ and $M_{\alpha'} = M_f M_p^{k+r} M_i$. For large $k$ one has

$$M_p^k = \Lambda_p^k P_u + \Lambda_p^{-k} P_s \sim \Lambda_p^k P_u \quad \text{as} \quad k \to \infty$$

It follows that

$$A_{\alpha'} \approx A_\alpha |\Lambda_p|^{-r/2}$$

and

$$\nu_{\alpha'} = \nu_\alpha + r \mu_p$$

Now one has all ingredients to calculate the semiclassical contribution of the trajectories
The semiclassical contribution

One replaces the sum over trajectories by a phase space integral

$$
\sum_{\alpha, \alpha'(a \rightarrow b)} |A_\alpha|^2 \ldots \approx \int dT \, ds \, du \, w_{p,T}(s, u) \, e^{-\mu T_{\text{exp}}} ,
$$

where $T_{\text{exp}} = T - t_{\text{enc}}^p$, $w_{p,T}(s, u) = \int dt_1 \frac{1}{k \Omega}$ and $k = t_{\text{enc}}^p / T_p$.

One finds again a factorization into contribution from links and the encounter. For the contribution of the periodic orbit encounter one needs

$$
\int ds \, du \, \frac{e^{\frac{i}{\hbar} s u (1 - \Lambda_p^{-r})}}{\Omega t_{\text{enc}}^p} e^{i \epsilon \mu t_{\text{enc}}^p} = \frac{i \epsilon \mu}{T_H |1 - \Lambda_p^{-r}|}
$$

This integral sums over all trajectories $\alpha$ with an arbitrary number of $k$ iterations of the periodic orbit. The semiclassical contribution to the integral comes from the vicinity of the origin where $k \rightarrow \infty$. 
The semiclassical contribution

The amplitude of the periodic orbit is obtained by using

$$\frac{1}{|\Lambda_p|^{r/2} |1 - \Lambda_p^{-r}|} = \frac{1}{\sqrt{|\text{det}(M_p^r - 1)|}}$$

Altogether one obtains the following diagrammatic rule for the encounter with the periodic orbit

$$2i\epsilon\mu A_{p,r} \cos \left( -\frac{1}{\hbar} r S_p + \frac{\pi}{2} r \mu_p + \frac{\epsilon\mu}{2} r T_p \right)$$

This yields the correct periodic orbit contribution to the time delay for systems without TRS. However, for systems with TRS the prefactor is slightly wrong. It contains one factor of $(M + 1)$ instead of $M$. 

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Periodic orbit encounters plus self-encounters

One has to consider also combinations of periodic orbit encounters and self-encounters. There are two different cases

- Periodic orbit encounters and self-encounters are separated from each other. These cases can be calculated by using the three diagrammatic rules that have been obtained before.

- Periodic-orbit encounters and self-encounters overlap. In other words, a self-encounter happens to occur in the close vicinity of a periodic orbit. This leads to interesting consequences. The simplest case is that of a two-encounter near a periodic orbit $p$ in systems with TRS.
A two-encounter near a periodic orbit $\rho$

In contrast to a usual self-encounter a trajectory $\alpha$ has many partners $\alpha'$. They can differ in the number of periodic orbit traversals before and after the loop, as long as the total number is the same as for $\alpha$.

If $\alpha$ has $k_1$ and $k_2$ periodic orbit traversals before and after the loop, then $\alpha'$ can have $k_1 + d$ and $k_2 - d$ traversals.

The number of “squares” belonging to the same partner orbit is

$$k \approx \frac{t_{p,\sigma}^p}{T_p}, \quad t_{enc}^{p,\sigma} = \frac{1}{\lambda_p T_p} \ln \frac{c^2}{\max_i |s_i| \times \max_j |u_j|}$$
An $l$-encounter near a periodic orbit $p$

An $l$-encounter is characterized by a permutation matrix $\pi$ which describes the reconnection of the links in the encounter region. We are interested in trajectories that have additional periodic orbit traversals $r_1, \ldots, r_l$ (whose sum is $r$) during the $l$ encounters with the periodic orbit.

$$\Delta S = s^T Bu - rS_p$$

where $B_{ji} = \delta_{ji} - \delta_{i(\pi(j))} \Lambda_p^{-r_j}$. One has $\det B = 1 - \Lambda_p^r$.

The resulting diagrammatic rule for the joint encounter is

$$2i\ell \epsilon \mu A_{p,r} \cos \left( -\frac{1}{\hbar}rS_p + \frac{\pi}{2}r\mu_p + \frac{\epsilon \mu}{2}rT_p \right)$$

Summing up all contributions one obtains the correct periodic orbit terms for the time delay in systems with or without TRS.
Conclusions

- The periodic orbit terms were obtained from trajectories that approach a periodic orbit very closely.
- For systems without TRS periodic encounters are sufficient, but for systems with TRS one needs to consider also combinations of periodic orbit encounters and self-encounters.
- The vicinity of periodic orbits leads to a rich variety of possible correlations between trajectories, not all of which have been explored. It is probable that they are relevant also in other contexts and deserve further study.
- The present calculation does not give periodic orbit terms for the Landauer-Büttiker conductance, because the conductance does not involve an energy difference.