Periodic orbit encounters: a mechanism for trajectory correlations

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Wigner time delay

$$\tau_{\mathrm{W}}(E) = -\frac{\mathrm{i}\hbar}{M}\mathrm{Tr}\left[S^\dagger(E)\frac{\mathrm{d}}{\mathrm{d}E}S(E)\right]$$

with scattering matrix $S(E)$ and M open scattering channels

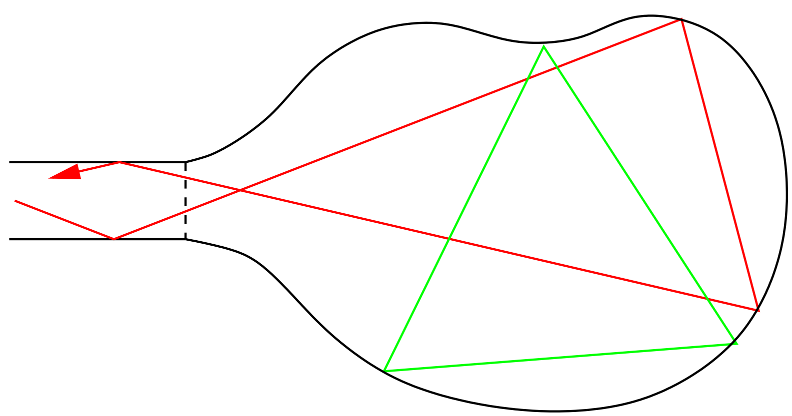
Relation to the 'density of states' (Friedel (1952))

$$\tau_{\mathrm{W}}(E) = \frac{2\pi\hbar}{M}d(E) \approx \frac{2\pi\hbar}{M}\left(\bar{d}(E) + d^{\mathrm{osc}}(E)\right)$$

The mean time delay is

$$\bar{\tau}_{\mathrm{W}}(E) \approx \frac{1}{\mu}$$

where μ is the classical escape rate



The two semiclassical formulas

The semiclassical formula for the elements of the scattering matrix is

$$S_{ba}(E) \approx \frac{1}{\sqrt{T_{\mathrm{H}}}} \sum_{\alpha(a\to b)} A_\alpha \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_\alpha} \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\nu_\alpha}$$

where $T_{\mathrm{H}} = 2\pi\hbar\bar{d}$ is the Heisenberg time

We arrive at two different semiclassical formulae for the Wigner time delay, and we expect the following to hold

$$\frac{1}{MT_{\mathrm{H}}} \sum_{a,b} \sum_{\alpha,\alpha'(a\to b)} T_\alpha A_\alpha A^*_{\alpha'} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_\alpha - S_{\alpha'})} \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}(\nu_\alpha - \nu_{\alpha'})}$$

$$\approx \bar{\tau}_{\mathrm{W}} + \frac{2}{M} \mathrm{Re} \sum_{p,r} A_{p,r}(E) \mathrm{e}^{\frac{\mathrm{i}}{\hbar} r S_p(E)} \mathrm{e}^{-\frac{\mathrm{i}\pi}{2} r\mu_p}$$

We will start from the double sum over scattering trajectories and derive all terms in the periodic orbit formula

The correlation function $C(\epsilon)$

Instead of working with the time delay, it is convenient to consider instead the following correlation function

$$C(\epsilon) = \sum_{a,b} S_{ba}\left(E + \frac{\epsilon\mu\hbar}{2}\right) S^*_{ba}\left(E - \frac{\epsilon\mu\hbar}{2}\right)$$

where μ is the classical escape rate. Using the unitarity of the scattering matrix, one obtains the Wigner time delay as

$$\tau_{\mathrm{W}} = \frac{-\mathrm{i}}{\mu M}\frac{\mathrm{d}}{\mathrm{d}\epsilon}C(\epsilon)\Big|_{\epsilon=0}$$

The semiclassical approximation for $C(\epsilon)$ is very similar to that of the Landauer-Büttiker conductance which is proportional to

$$\mathcal{T} = \sum_{b_{\mathrm{out}}a_{\mathrm{in}}} \left\langle S_{b_{\mathrm{out}}a_{\mathrm{in}}}(E) S^*_{b_{\mathrm{out}}a_{\mathrm{in}}}(E)\right\rangle$$

The diagonal approximation

A trajectory is paired only with itself (or its time-reverse). One uses a sum rule for open trajectories which is based on the ergodic exploration of the available phase space plus the finite escape probability

$$\sum_{\alpha(a\to b)} |A_\alpha|^2 \mathrm{e}^{\mathrm{i}\epsilon\mu T_\alpha} \approx \int_0^\infty \mathrm{d}T\, \mathrm{e}^{-\mu T}\, \mathrm{e}^{\mathrm{i}\epsilon\mu T}$$

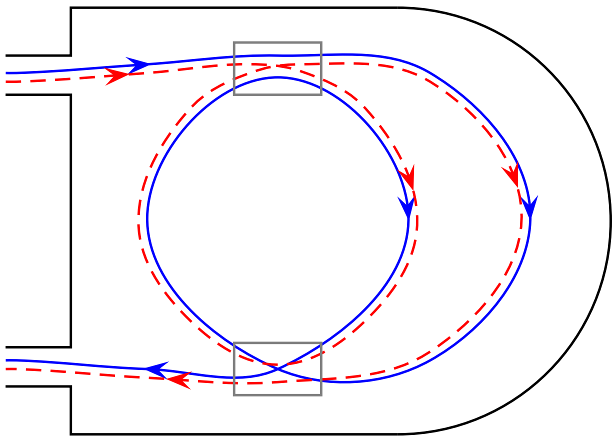
The sum over channels gives a factor M^2 for systems without TRS ($\kappa = 1$), and a factor of $M(M+1)$ for systems with TRS ($\kappa = 2$), because one can pair a trajectory with its time-reverse if $a = b$

$$C^{\mathrm{diag}}(\epsilon) \approx \frac{(M+\kappa-1)}{(1-\mathrm{i}\epsilon)}$$

This yields the correct mean time delay μ^{-1} for systems without TRS, but it is slightly wrong for systems with TRS (the numerator should be M)

Off-diagonal terms for $\overline{\tau_{\mathrm{W}}}$

The off-diagonal contributions come from trajectories with self-encounters and their partner orbits. Similar to conductance (Richter, M.S.(2002); Heusler et al (2006); Müller et al (2007)).



The number of l-encounters are collected in a vector $\boldsymbol{v}$.

Diagrammatic rules:

For each link: $[M\,(1-\mathrm{i}\epsilon)]^{-1}$

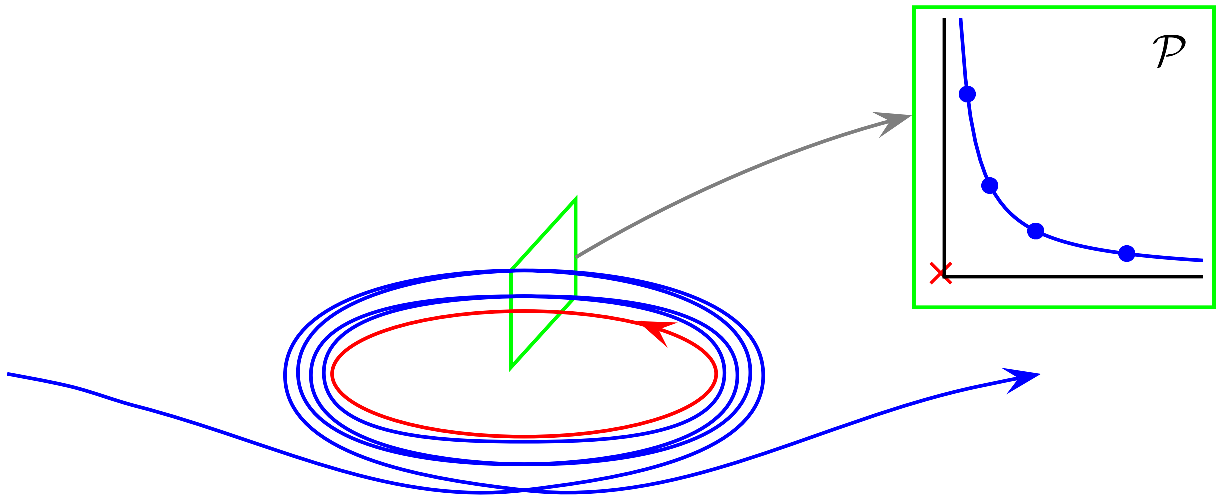
For each l-encounter: $-M\,(1-\mathrm{i}l\epsilon)$

With a sum rule for the number of *structures* $N(v)$ one arrives at

$$\bar{C}(\epsilon) \approx M\,[1+\mathrm{i}\epsilon+O(\epsilon^2)]$$

Periodic orbit encounters

For the periodic orbit contributions we consider trajectories that approach a periodic orbit p, follow it a number of times, and leave it again



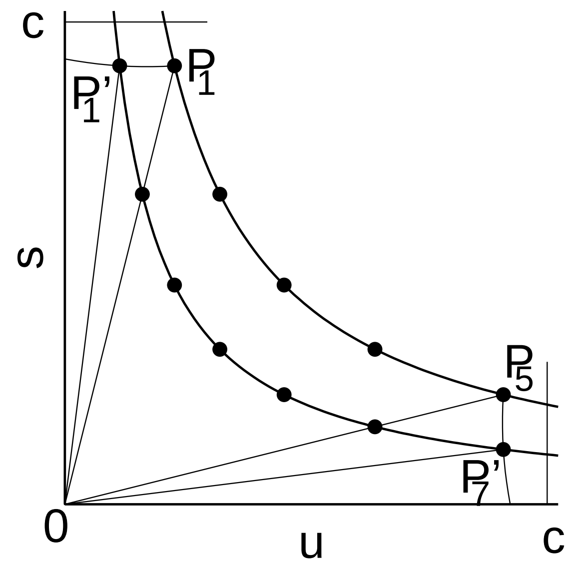
The Poincaré map has a simple form in the vicinity of p

$$s' = \Lambda_p^{-1}\, s\,, \qquad u' = \Lambda_p\, u$$

where $\Lambda_p = \pm e^{\lambda_p T_p}$ is an eigenvalue of the stability matrix M_p.

The trajectory pairs

Consider an orbit that has k intersections in the Poincaré surface, $P_1, \ldots, P_k$, limited by the constant c. Its partner orbit has r more intersections



$$P_1 = (u_1, s_1) \implies P_1' \approx (\Lambda_p^{-r} u_1, s_1)$$

The action difference is

$$\Delta S = S_\alpha - S_{\alpha'} = su(1 - \Lambda_p^{-r}) - rS_p$$

One can define an encounter time for α which is given by

$$t^p_{\rm enc}(s,u) = k\,T_p \approx \frac{1}{\lambda_p} \ln \frac{c^2}{|us|}$$

The semiclassical contribution

The semiclassical amplitudes are proportional to the M_{12}-element of the stability matrix. We can write $M_\alpha = M_f M_p^k M_i$ and $M_{\alpha'} = M_f M_p^{k+r} M_i$. For large k one has

$$M_p^k = \Lambda_p^k P_u + \Lambda_p^{-k} P_s \sim \Lambda_p^k P_u \quad \text{as} \quad k \to \infty$$

It follows that

$$A_{\alpha'} \approx A_\alpha \, |\Lambda_p|^{-r/2}$$

and

$$\nu_{\alpha'} = \nu_\alpha + r\mu_p$$

Now one has all ingredients to calculate the semiclassical contribution of the trajectories

The semiclassical contribution

One replaces the sum over trajectories by a phase space integral

$$\sum_{\alpha,\alpha'(a\to b)} |A_\alpha|^2 \ldots \approx \int \mathrm{d}T\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{u}\; w_{p,T}(\boldsymbol{s},\boldsymbol{u})\,\mathrm{e}^{-\mu T_{\mathrm{exp}}}\;,$$

where $T_{\mathrm{exp}} = T - t_{\mathrm{enc}}^p$, $w_{p,T}(s,u) = \int \mathrm{d}t_1 \frac{1}{k\,\Omega}$ and $k = t_{\mathrm{enc}}^p / T_p$.

One finds again a factorization into contribution from links and the encounter. For the contribution of the periodic orbit encounter one needs

$$\int \mathrm{d}s\,\mathrm{d}u\,\frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}su(1-\Lambda_p^{-r})}\,\mathrm{e}^{\mathrm{i}\epsilon\mu t_{\mathrm{enc}}^p}}{\Omega\, t_{\mathrm{enc}}^p} = \frac{\mathrm{i}\epsilon\mu}{T_{\mathrm{H}}\,|1-\Lambda_p^{-r}|}$$

This integral sums over all trajectories α with an arbitrary number of k iterations of the periodic orbit. The semiclassical contribution to the integral comes from the vicinity of the origin where $k \to \infty$.

The semiclassical contribution

The amplitude of the periodic orbit is obtained by using

$$\frac{1}{|\Lambda_p|^{r/2}\,|1-\Lambda_p^{-r}|} = \frac{1}{\sqrt{|\det(M_p^r - 1)|}}$$

Altogether one obtains the following diagrammatic rule for the encounter with the periodic orbit

$$2\mathrm{i}\epsilon\mu A_{p,r}\cos\left(-\frac{1}{\hbar}rS_p + \frac{\pi}{2}r\mu_p + \frac{\epsilon\mu}{2}rT_p\right)$$

This yields the correct periodic orbit contribution to the time delay for systems without TRS. However, for systems with TRS the prefactor is slightly wrong. It contains one factor of $(M+1)$ instead of M.

Periodic orbit encounters plus self-encounters

One has to consider also combinations of periodic orbit encounters and self-encounters. There are two different cases

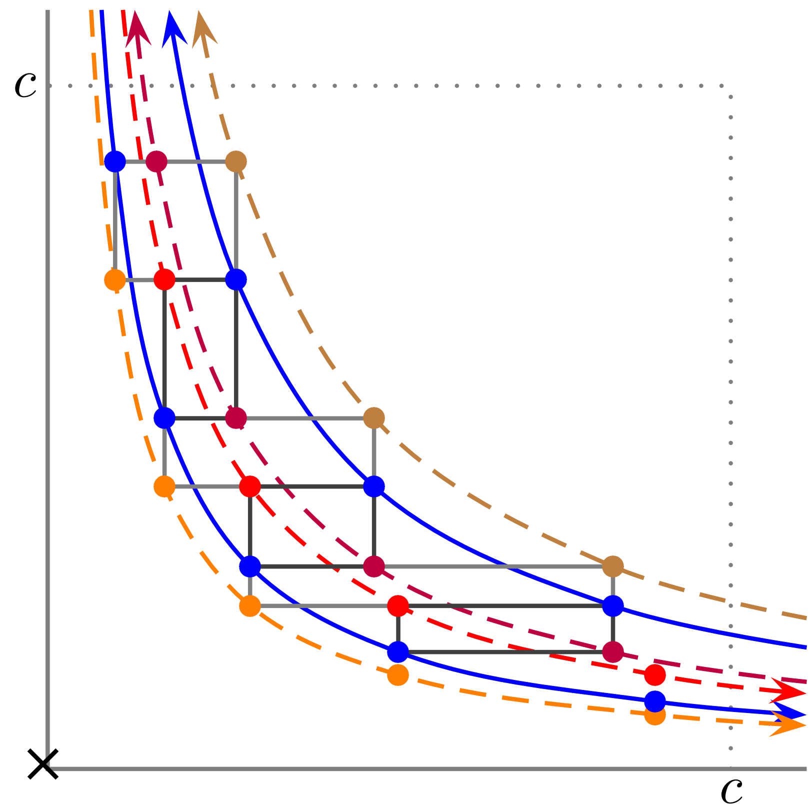
- Periodic orbit encounters and self-encounters are separated from each other. These cases can be calculated by using the three diagrammatic rules that have been obtained before.

- Periodic-orbit encounters and self-encounters overlap. In other words, a self-encounter happens to occur in the close vicinity of a periodic orbit. This leads to interesting consequences. The simplest case is that of a two-encounter near a periodic orbit p in systems with TRS.

A two-encounter near a periodic orbit p

In contrast to a usual self-encounter a trajectory α has *many* partners α'. They can differ in the number of periodic orbit traversals before and after the loop, as long as the total number is the same as for α.

If α has k_1 and k_2 periodic orbit traversals before and after the loop, then α' can have $k_1 + d$ and $k_2 - d$ traversals



The number of “squares” belonging to the same partner orbit is

$$k \approx t_{\rm enc}^{p,\sigma}/T_p\,, \quad t_{\rm enc}^{p,\sigma} = \frac{1}{\lambda_p\,T_p}\ln\frac{c^2}{\max_i|s_i| \times \max_j|u_j|}$$

An l-encounter near a periodic orbit p

An l-encounter is characterized by a permutation matrix π which describes the reconnection of the links in the encounter region. We are interested in trajectories that have additional periodic orbit traversals $r_1, \ldots, r_l$ (whose sum is r) during the l encounters with the periodic orbit.

$$\Delta S = \boldsymbol{s}^{\mathrm{T}} B \boldsymbol{u} - r S_p$$

where $B_{ji} = \delta_{ji} - \delta_{i\pi(j)} \Lambda_p^{-r_j}$. One has $\det B = 1 - \Lambda_p^r$.

The resulting diagrammatic rule for the joint encounter is

$$2\mathrm{i} l \epsilon \mu A_{p,r} \cos\left(-\frac{1}{\hbar} r S_p + \frac{\pi}{2} r \mu_p + \frac{\epsilon\mu}{2} r T_p \right)$$

Summing up all contributions one obtains the correct periodic orbit terms for the time delay in systems with or without TRS.

Conclusions

- The periodic orbit terms were obtained from trajectories that approach a periodic orbit very closely
- For systems without TRS periodic encounters are sufficient, but for systems with TRS one needs to consider also combinations of periodic orbit encounters and self-encounters
- The vicinity of periodic orbits leads to a rich variety of possible correlations between trajectories, not all of which have been explored. It is probable that they are relevant also in other contexts and deserve further study
- The present calculation does not give periodic orbit terms for the Landauer-Büttiker conductance, because the conductance does not involve an energy difference