Extreme statistics of random and quantum chaotic states

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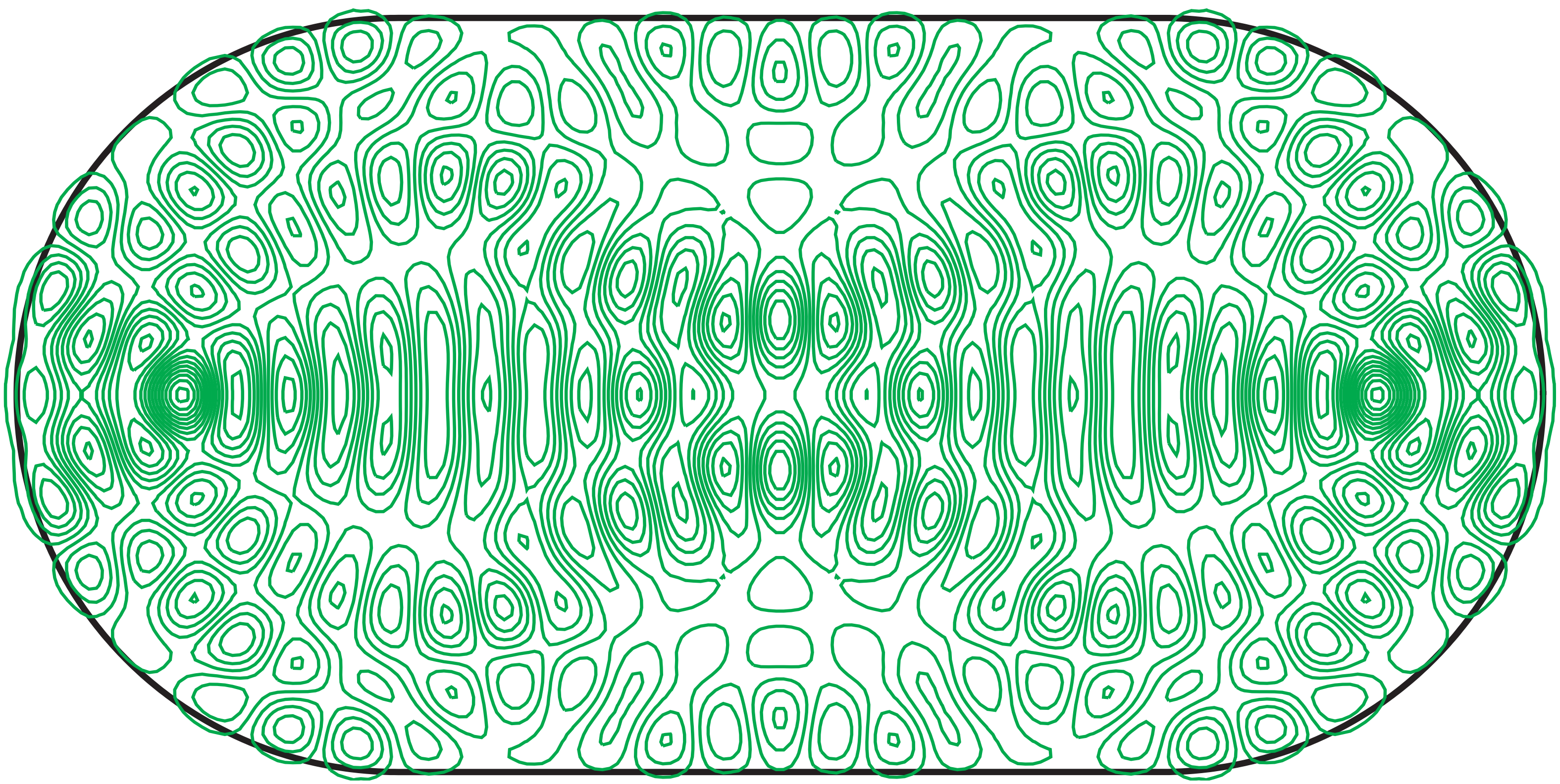

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Today's Thread of Logic

1) The statistics of extreme eigenstate intensities

- Densities and distribution functions

    Largest and smallest intensities

- Universal distributions

    Weibull/Fréchet

    Gumbel

2) Random states and chaotic quantum eigenstates

- Complex random states and unitary ensembles[1]

    Exact results

    Kicked rotor

- Real random states and orthogonal ensembles

    Saddle point approximations

[1] recent published work: A. Lakshminarayan et al., *Phys. Rev. Lett.* **100**, 044103 (2008).

Distribution functions for maxima and minima intensities

- Suppose an ensemble of systems acts in an N-dimensional vector space, $\{|j\rangle\}$, $j = 1, ..., N$ with eigenvectors of a member system, $\{|\phi_n\rangle\}$, $n = 1, ..., N$. Then, the intensities for a single eigenstate are

$$s_j = |\langle\phi_n|j\rangle|^2$$

- Using n and the system ensemble, a joint intensity probability density can be defined and denoted

$$\rho(\vec{s}; N) = \rho(s_1, s_2, ..., s_N; N)$$

- Let the maximum intensity be $s = \max[s_j]$, $j = 1, ...N$
- The distribution function is given by,

$$F_{max}(t; N) = \int_{\frac{1}{N}}^{t} \mathrm{d}s \, \rho_{max}(s; N) = \int_{0}^{t} \mathrm{d}\vec{s} \, \rho(\vec{s}; N) \, ; \quad \left[\frac{\mathrm{d}}{\mathrm{d}t} F_{max}(t; N) = \rho_{max}(t; N)\right]$$

- Or for the minimum intensity $s = \min[s_j]$, $j = 1, ...N$

$$F_{min}(t; N) = 1 - \int_{t}^{\frac{1}{N}} \mathrm{d}s \, \rho_{min}(s; N) = 1 - \int_{t}^{1} \mathrm{d}\vec{s} \, \rho(\vec{s}; N)$$

Distribution functions for maxima and minima
uncorrelated variables

- Let the $\{s_j\}$, $j = 1, ..., N$ be similarly distributed independent random variables.
- A joint probability density can be defined and denoted

$$\rho(\vec{s}; N) = \rho(s_1, s_2, ..., s_N; N) = \prod_{j=1}^{N} \rho(s_j; N)$$

- The distribution function of the maximum is given by,

$$F_{max}(t; N) = \int^{t} \mathrm{d}\vec{s}\, \rho(\vec{s}; N) = \left[\int^{t} \mathrm{d}s_j\ \rho(s_j; N)\right]^{N} \quad \text{(any j)}$$

- Or for the minimum

$$F_{min}(t; N) = 1 - \int_{t} \mathrm{d}\vec{s}\, \rho(\vec{s}; N) = 1 - \left[1 - \int^{t} \mathrm{d}s_j\ \rho(s_j; N)\right]^{N}$$

Universal distribution functions - Fisher/Tippett 1928
uncorrelated variables (cont.)

- The Weibull/Fréchet distribution function

$$F(t;N) = 1 - \exp[-(\pm t - a_N)^{\gamma_N}/b_N]$$

  is expected for uncorrelated random variables with compact support from above or below (or heavy tailed densities).

- The Gumbel distribution function

$$F(t;N) = \exp[-\mathrm{e}^{-(t-a_N)/b_N}]$$

  is expected for uncorrelated random variables with non-compact support whose tails decay at least exponentially fast.

- For example, consider the uniform density $\rho(t) = 1$ $(0 \leq t \leq 1)$:

$$\begin{array}{lclll} F_{max}(t;N) & = & t^N & \longrightarrow \mathrm{e}^{-N(1-t)} & \text{Weibull} \\ F_{min}(t;N) & = & 1-(1-t)^N & \longrightarrow 1-\mathrm{e}^{-Nt} & \text{Fréchet} \end{array}$$

Two relevant examples: complex and real Gaussian amplitudes
uncorrelated variables (cont.)

- Complex Gaussian amplitude leads to $\frac{1}{N}$-mean intensity density:

$$\rho(t) = N\mathrm{e}^{-Nt} \qquad (0 \leq t \leq \infty)$$

- and hence

$$F_{max}(t;N) = \left(1-\mathrm{e}^{-Nt}\right)^N \rightarrow \exp\left(-\mathrm{e}^{-N(t-\frac{1}{N}\ln N)}\right) \quad \text{Gumbel}$$

$$F_{min}(t;N) = 1-\left[1-\left(1-\mathrm{e}^{-Nt}\right)\right]^N = 1-\mathrm{e}^{-N^2t} \quad \text{Fréchet}$$

- Real Gaussian amplitude leads to $\frac{1}{N}$-mean intensity density:

$$\rho(t) = \sqrt{\frac{N}{2\pi t}}\mathrm{e}^{-Nt/2} \qquad (0 \leq t \leq \infty)$$

- and hence

$$F_{max}(t;N) = \mathrm{erf}^N\left(\sqrt{Nt/2}\right) \rightarrow \exp\left(-\mathrm{e}^{-\frac{N}{2}(t-\frac{1}{N}\ln\frac{2N}{\pi t})}\right) \quad \text{Gumbel?}$$

$$F_{min}(t;N) = 1-\left[1-\mathrm{erf}\left(\sqrt{Nt/2}\right)\right]^N \rightarrow 1-\mathrm{e}^{-\sqrt{\frac{2N^3t}{\pi}}} \quad \text{Fréchet}$$

Joint probability densities for intensities
correlated variables

- A norm constraint is naturally expressed in amplitude variables:

$$\rho_\beta(z_1, z_2, \ldots, z_N) = \frac{\Gamma\left(\frac{N\beta}{2}\right)}{\pi^{N\beta/2}}\, \delta\left(\sum_{j=1}^{N} |z_j|^2 - 1\right)$$

  where $\beta = 1, 2$ for real and complex respectively. The real case corresponds to the orthogonal random matrix ensembles and the complex case to the unitary ensembles. Switching to intensities:

$$\rho_\beta(\vec{s}; N) = \pi^{N(\beta/2-1)}\Gamma\left(\frac{N\beta}{2}\right)\left[\prod_{j=1}^{N} s_j^{\beta/2-1}\mathrm{d}s_j\right]\delta\left(\sum_{j=1}^{N} s_j - 1\right)$$

- The complex case is equivalent to the "broken stick problem" in which $N-1$ cuts at uniformly random locations are made in a unit length stick.

- The real case is intimately connected to the relationship between hyperspherical and cartesian coordinates.

An auxiliary function for "decorrelating" intensities

- The distribution function for the maximum is:

$$F^{\beta}_{max}(t;N) = \pi^{N(\beta/2-1)}\Gamma\left(\frac{N\beta}{2}\right)\left[\prod_{j=1}^{N}\int_0^t s_j^{\beta/2-1}\mathrm{d}s_j\right]\delta\left(\sum_{j=1}^{N} s_j \; - \; 1\right)$$

- Define the auxiliary function $G^{\beta}(t,u;N)$ which results from replacing unity in the norm constraint by u and thus, $F^{\beta}_{max}(t;N) = G^{\beta}(t,u=1;N)$.

- The Laplace transform of $G^{\beta}(t,u;N)$ renders the integrals over the N differentials $\mathrm{d}s_j$ into a product form and gives:

$$\int_0^{\infty} e^{-us} G_{\beta}(t,N,u)du = \begin{cases} \Gamma(\frac{N}{2})\left(\frac{\mathrm{erf}(\sqrt{st})}{\sqrt{s}}\right)^N & \text{real} \\ \Gamma(N)\left(\frac{1-e^{-st}}{s}\right)^N & \text{complex} \end{cases}$$

- The N integrals have been performed at the cost of now needing the inverse Laplace transforms of these expressions.

Exact results for unitary ensembles

- The distribution function for the maximum follows by expanding the N^{th} power and using the inverse Laplace transform:

$$\mathcal{L}_s^{-1}\left(\frac{e^{-smt}}{s^N}\right) = \frac{1}{\Gamma(N)}(u-mt)^{N-1}\Theta(u-mt)$$

  and gives

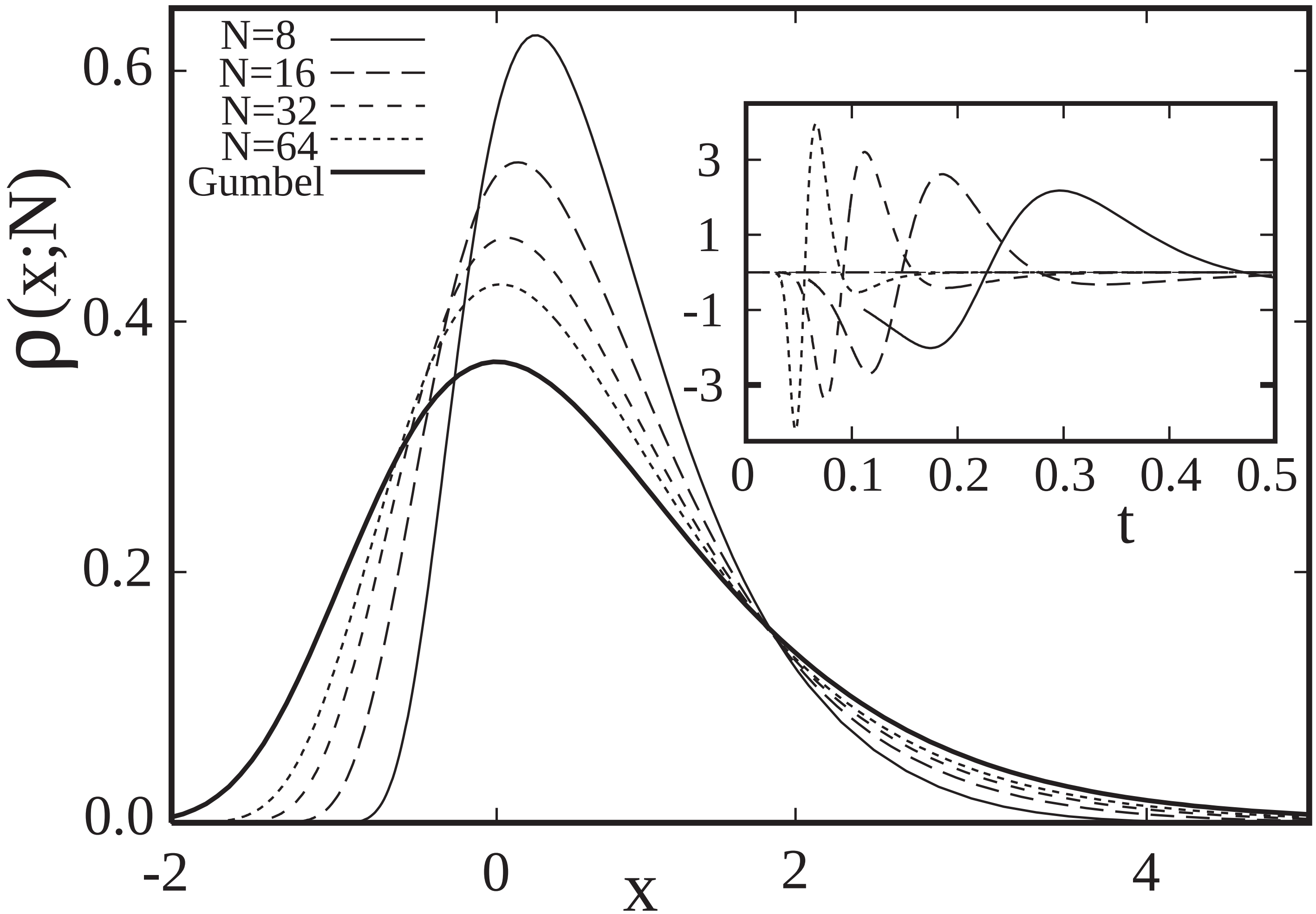
$$F_{max}^{\beta=2}(t;N) = \sum_{m=0}^{N}\binom{N}{m}(-1)^m(1-mt)^{N-1}\Theta(1-mt)$$

- Interestingly, this reduces to a piecewise smooth expression with the intervals $I_k = [1/(k+1), 1/k]$, where $k = 1, 2, \cdots, N-1$

$$F_{max}(t \in I_k; N) = \sum_{m=0}^{k}\binom{N}{m}(-1)^m(1-mt)^{N-1}$$

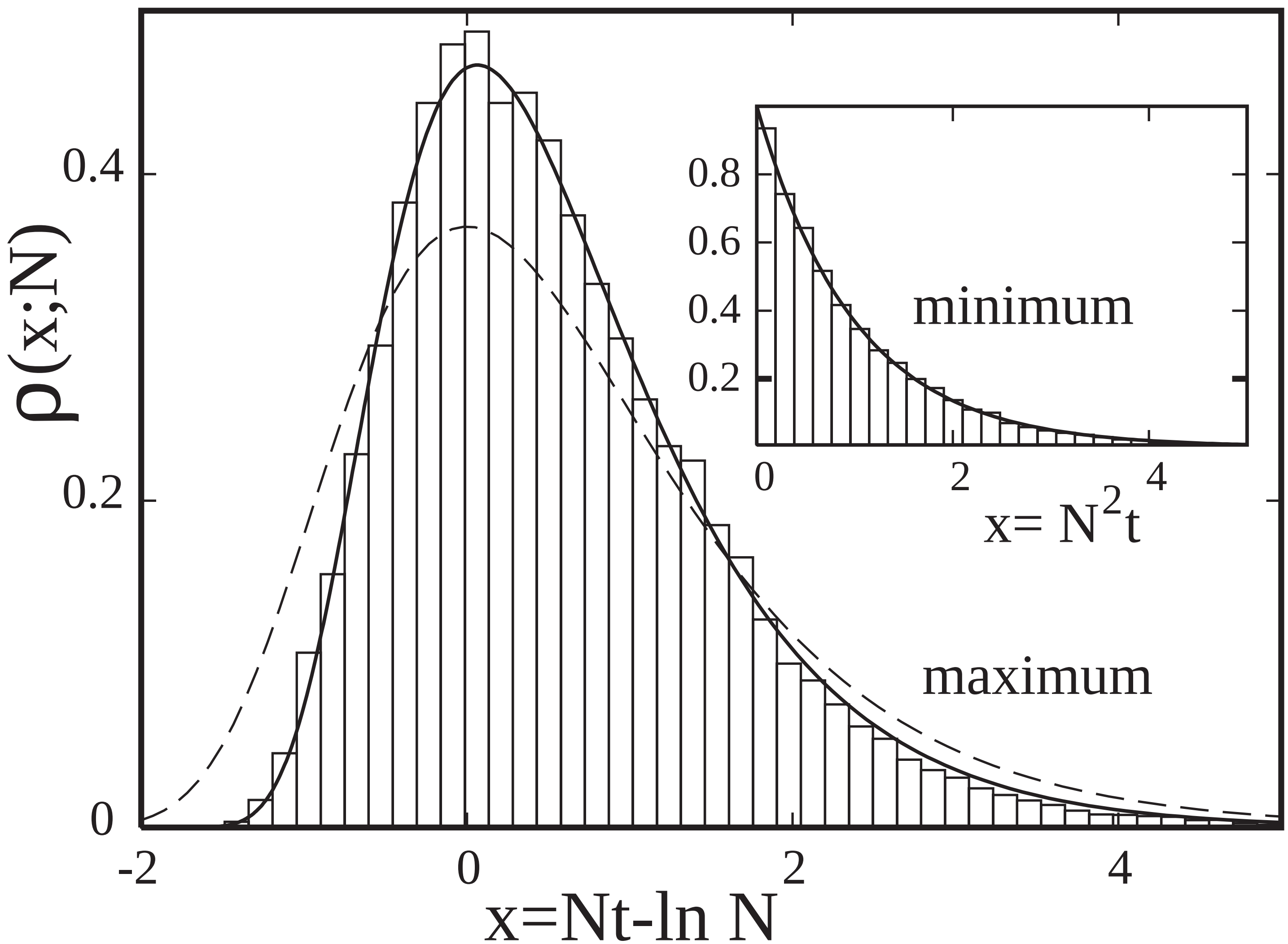
- All distributions resulting from correlated variables possessing at least unit norm constraints, satisfy a combinatoric form of this type.

Exact results for unitary ensembles (cont.)



The exact probability density and the asymptotic Gumbel density using the scaled variable $x = N(t - \ln(N)/N)$ with increasing N. The inset shows the difference between the exact and the Gumbel densities for the same values of N, but in the unscaled variable.

The quantum kicked rotor as an example of a chaotic system



The probability densities (histograms) of the scaled maximum and minimum (inset) intensity of eigenfunctions in the position basis of the quantum kicked rotor for $N = 32$ in the parameter range $13.8 < K < 14.8$. Shown as a continuous line is the exact density for random states while the dotted ones are the respective Gumbel and Fréchet densities.

A saddle point approximation for the orthogonal ensembles

A saddle point approximation for the inverse Laplace transform:

$$\mathcal{L}_s^{-1}\left((\pi st)^{m/2}\,\mathrm{e}^{mst}\mathrm{erfc}^m(\sqrt{st})\left(\tfrac{1}{\pi t}\right)^{m/2}\frac{e^{-smt}}{s^{\frac{N+m}{2}}}\right)\approx\left(\tfrac{N+m}{2-2mt}\right)^{m/2}$$

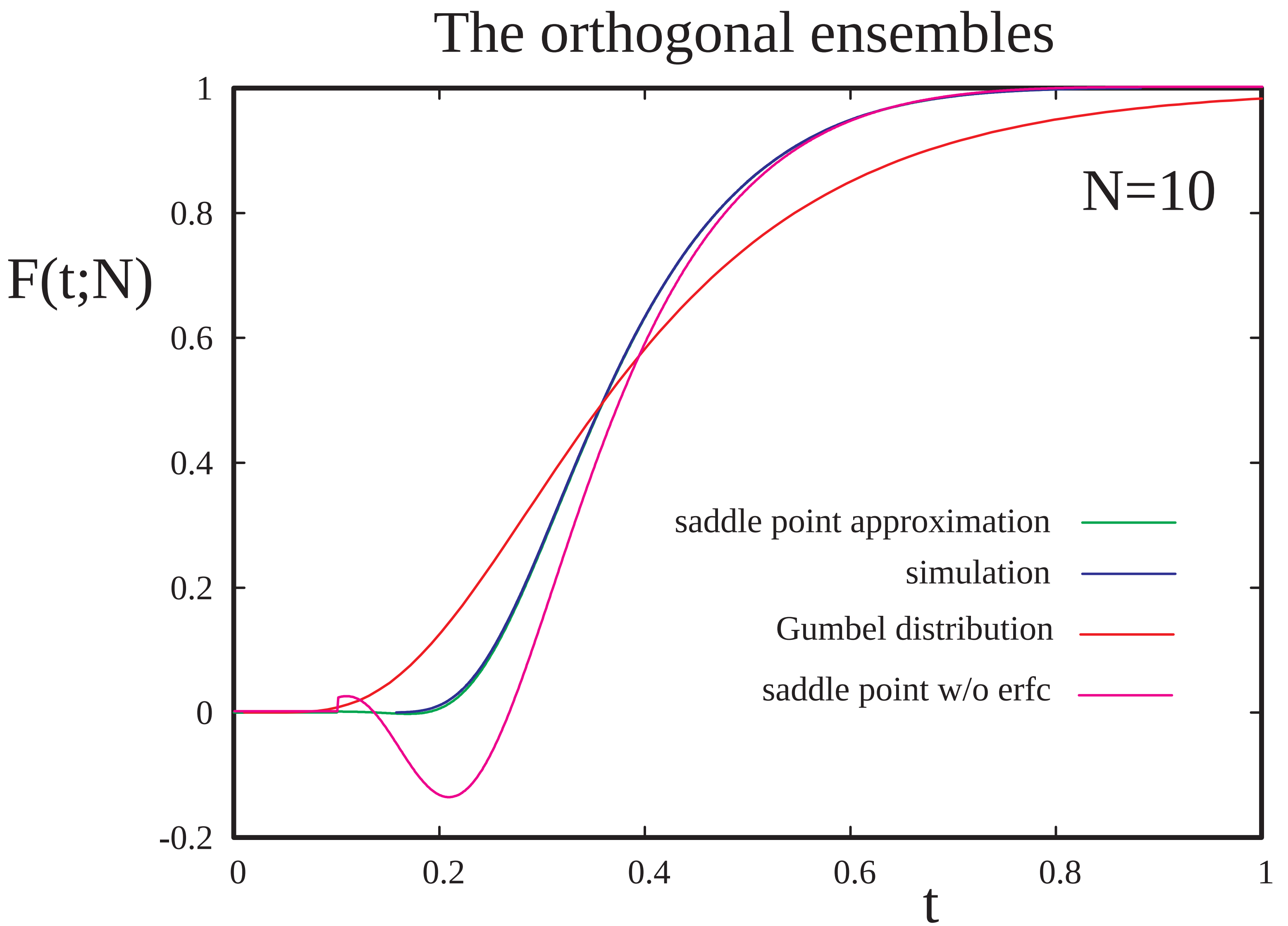
$$\times\exp\left(\tfrac{m(N+m)t}{2-2mt}\right)\mathrm{erfc}^{\mathrm{m}}\left(\sqrt{\tfrac{(\mathrm{N+m})\mathrm{t}}{2-2\mathrm{mt}}}\right)\frac{1}{\Gamma\left(\frac{N+m}{2}\right)}(1-mt)^{\frac{N+m}{2}-1}\Theta(1-mt)$$

gives $F_{max}^{\beta=1}(t;N)=$

$$\sum_{m=0}^{k}\frac{\binom{N}{m}(-1)^m\left(\frac{N+m}{2}\right)^{\frac{m}{2}}\Gamma\left(\frac{N}{2}\right)(1-mt)^{\frac{N}{2}-1}}{\Gamma\left(\frac{N+m}{2}\right)}\exp\left(\frac{m\left(\frac{N+m}{2}\right)t}{1-mt}\right)\mathrm{erfc}^{\mathrm{m}}\left(\sqrt{\frac{\left(\frac{\mathrm{N+m}}{2}\right)\mathrm{t}}{1-\mathrm{mt}}}\right)$$

or more simply using only the asymptotic form of erfc(z),

$$F_{max}^{\beta=1}(t;N)=\sum_{m=0}^{k}\binom{N}{m}(-1)^m\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+m}{2}\right)}\frac{(1-mt)^{\frac{N+m}{2}-1}}{(\pi t)^{\frac{m}{2}}}$$



Comparison of the maximum intensity distribution functions for the orthogonal ensembles. The saddle point approximation improves considerably the agreement with the "exact" result from simulation vis-a-vis the asymptotic Gumbel form. The simpler form without the complementary error function is an improvement, but not good enough for small N to warrant its use.

Concluding remarks

- The statistical properties of extreme intensities has not previously been applied to understanding better the eigenstates of quantum systems.
- It is possible to derive some compact, exact results for the unitary ensembles with any dimensionality and give excellent approximations for the same quantities in the orthogonal ensemble.
- The maximum intensities tend to the infinite dimensional limit very slowly and thus the functional forms contain some information about system size. They further tend toward the Gumbel distribution although, a priori, one might have expected Weibull. Means scale as $\ln a_\beta N/N$ for unitary and (roughly for) orthogonal ensembles.
- The minimum intensity statistics tend much more rapidly toward their infinite dimensional Fréchet limiting form. The mean minima are N^{-2} and πN^{-3} for the unitary and orthogonal ensembles respectively.
- These "extreme" measures give us a new way to explore non-ergodic eigenstate behaviors.
  - It would be worthwhile exploring: i) other measures such as the intensity densities in the neighborhood of the maxima or minima, and ii) how system dynamics lead to deviations from the chaotic statistical extremes.