Extreme statistics of random and quantum chaotic states

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Today’s Thread of Logic

1) The statistics of extreme eigenstate intensities
   • Densities and distribution functions
     Largest and smallest intensities
   • Universal distributions
     Weibull/Fréchet
     Gumbel

2) Random states and chaotic quantum eigenstates
   • Complex random states and unitary ensembles\(^1\)
     Exact results
     Kicked rotor
   • Real random states and orthogonal ensembles
     Saddle point approximations

Distribution functions for maxima and minima intensities

- Suppose an ensemble of systems acts in an $N$-dimensional vector space, $\{|j\rangle\}$, $j = 1, ..., N$ with eigenvectors of a member system, $\{|\phi_n\rangle\}$, $n = 1, ..., N$. Then, the intensities for a single eigenstate are

$$s_j = |\langle \phi_n | j \rangle|^2$$

- Using $n$ and the system ensemble, a joint intensity probability density can be defined and denoted

$$\rho(\vec{s}; N) = \rho(s_1, s_2, ..., s_N; N)$$

- Let the maximum intensity be $s = \max [s_j]$, $j = 1, ... N$

- The distribution function is given by,

$$F_{max}(t; N) = \int_{\frac{1}{N}}^{t} ds \rho_{max}(s; N) = \int_{0}^{t} d\vec{s} \rho(\vec{s}; N) \quad \left[ \frac{d}{dt} F_{max}(t; N) = \rho_{max}(t; N) \right]$$

- Or for the minimum intensity $s = \min [s_j]$, $j = 1, ... N$

$$F_{min}(t; N) = 1 - \int_{t}^{\frac{1}{N}} ds \rho_{min}(s; N) = 1 - \int_{t}^{1} d\vec{s} \rho(\vec{s}; N)$$
Distribution functions for maxima and minima
uncorrelated variables

- Let the \( \{s_j\}, j = 1, \ldots, N \) be similarly distributed independent random variables.

- A joint probability density can be defined and denoted

\[
\rho(\vec{s}; N) = \rho(s_1, s_2, \ldots, s_N; N) = \prod_{j=1}^{N} \rho(s_j; N)
\]

- The distribution function of the maximum is given by,

\[
F_{\text{max}}(t; N) = \int_{\vec{s}} d\vec{s} \, \rho(\vec{s}; N) = \left[ \int_{s_j} d s_j \, \rho(s_j; N) \right]^N \quad (\text{any } j)
\]

- Or for the minimum

\[
F_{\text{min}}(t; N) = 1 - \int_{\vec{s}} d\vec{s} \, \rho(\vec{s}; N) = 1 - \left[ 1 - \int_{s_j} d s_j \, \rho(s_j; N) \right]^N
\]
• The Weibull/Fréchet distribution function

\[ F(t; N) = 1 - \exp[-(\pm t - a_N)^{\gamma_N} / b_N] \]

is expected for uncorrelated random variables with compact support from above or below (or heavy tailed densities).

• The Gumbel distribution function

\[ F(t; N) = \exp[-e^{-(t-a_N)/b_N}] \]

is expected for uncorrelated random variables with non-compact support whose tails decay at least exponentially fast.

• For example, consider the uniform density \( \rho(t) = 1 \) \((0 \leq t \leq 1)\):

\[
\begin{align*}
F_{\text{max}}(t; N) &= t^N \quad \longrightarrow e^{-N(1-t)} \quad \text{Weibull} \\
F_{\text{min}}(t; N) &= 1 - (1 - t)^N \quad \longrightarrow 1 - e^{-Nt} \quad \text{Fréchet}
\end{align*}
\]
Two relevant examples: complex and real Gaussian amplitudes uncorrelated variables (cont.)

- Complex Gaussian amplitude leads to $\frac{1}{N}$-mean intensity density:
  \[ \rho(t) = Ne^{-Nt} \quad (0 \leq t \leq \infty) \]

- and hence
  \[
  F_{\text{max}}(t; N) = (1 - e^{-Nt})^N \rightarrow \exp \left( -e^{-N(t - \frac{1}{N} \ln N)} \right) \quad \text{Gumbel}
  \]
  \[
  F_{\text{min}}(t; N) = 1 - \left[ 1 - (1 - e^{-Nt}) \right]^N = 1 - e^{-N^2t} \quad \text{Fréchet}
  \]

- Real Gaussian amplitude leads to $\frac{1}{N}$-mean intensity density:
  \[ \rho(t) = \sqrt{\frac{N}{2\pi t}} e^{-Nt/2} \quad (0 \leq t \leq \infty) \]

- and hence
  \[
  F_{\text{max}}(t; N) = \text{erf}^N \left( \sqrt{Nt/2} \right) \rightarrow \exp \left( -e^{-\frac{N}{2}(t - \frac{1}{N} \ln \frac{2N}{\pi t})} \right) \quad \text{Gumbel?}
  \]
  \[
  F_{\text{min}}(t; N) = 1 - \left[ 1 - \text{erf} \left( \sqrt{Nt/2} \right) \right]^N \rightarrow 1 - e^{-\sqrt{\frac{2N^3t}{\pi}}} \quad \text{Fréchet}
  \]
Joint probability densities for intensities
correlated variables

- A norm constraint is naturally expressed in amplitude variables:

\[
\rho_\beta(z_1, z_2, \ldots, z_N) = \frac{\Gamma\left(\frac{N\beta}{2}\right)}{\pi^{N\beta/2}} \delta\left(\sum_{j=1}^{N} |z_j|^2 - 1\right)
\]

where \( \beta = 1, 2 \) for real and complex respectively. The real case corresponds to the orthogonal random matrix ensembles and the complex case to the unitary ensembles. Switching to intensities:

\[
\rho_\beta(\vec{s}; N) = \pi^{N(\beta/2 - 1)} \Gamma\left(\frac{N\beta}{2}\right) \left[ \prod_{j=1}^{N} s_j^{\beta/2 - 1} ds_j \right] \delta\left(\sum_{j=1}^{N} s_j - 1\right)
\]

- The complex case is equivalent to the “broken stick problem” in which \( N - 1 \) cuts at uniformly random locations are made in a unit length stick.

- The real case is intimately connected to the relationship between hyperspherical and cartesian coordinates.
An auxiliary function for “decorrelating” intensities

- The distribution function for the maximum is:

\[
F_{\beta}^{\text{max}}(t; N) = \pi^{N(\beta/2-1)} \Gamma \left( \frac{N\beta}{2} \right) \left[ \prod_{j=1}^{N} \int_{0}^{t} s_j^{\beta/2-1} ds_j \right] \delta \left( \sum_{j=1}^{N} s_j - 1 \right)
\]

- Define the auxiliary function \( G_{\beta}(t, u; N) \) which results from replacing unity in the norm constraint by \( u \) and thus, \( F_{\beta}^{\text{max}}(t; N) = G_{\beta}(t, u = 1; N) \).

- The Laplace transform of \( G_{\beta}(t, u; N) \) renders the integrals over the \( N \) differentials \( ds_j \) into a product form and gives:

\[
\int_{0}^{\infty} e^{-us} G_{\beta}(t, N, u) du = \begin{cases} 
\Gamma \left( \frac{N}{2} \right) \left( \frac{\text{erf}(\sqrt{st})}{\sqrt{s}} \right)^{N} & \text{real} \\
\Gamma(N) \left( \frac{1-e^{-st}}{s} \right)^{N} & \text{complex}
\end{cases}
\]

- The \( N \) integrals have been performed at the cost of now needing the inverse Laplace transforms of these expressions.
Exact results for unitary ensembles

• The distribution function for the maximum follows by expanding the $N^{th}$ power and using the inverse Laplace transform:

$$ \mathcal{L}_{s}^{-1}\left(\frac{e^{-smt}}{s^{N}}\right) = \frac{1}{\Gamma(N)}(u - mt)^{N-1}\Theta(u - mt)$$

and gives

$$ F_{\beta=2}^{\beta=2}(t; N) = \sum_{m=0}^{N} \binom{N}{m}(-1)^{m}(1 - mt)^{N-1}\Theta(1 - mt)$$

• Interestingly, this reduces to a piecewise smooth expression with the intervals $I_{k} = [1/(k + 1), 1/k]$, where $k = 1, 2, \cdots, N - 1$

$$ F_{\text{max}}(t \in I_{k}; N) = \sum_{m=0}^{k} \binom{N}{m}(-1)^{m}(1 - mt)^{N-1}$$

• All distributions resulting from correlated variables possessing at least unit norm constraints, satisfy a combinatoric form of this type.
Exact results for unitary ensembles (cont.)

The exact probability density and the asymptotic Gumbel density using the scaled variable \( x = N(t - \ln(N)/N) \) with increasing \( N \). The inset shows the difference between the exact and the Gumbel densities for the same values of \( N \), but in the unscaled variable.
The probability densities (histograms) of the scaled maximum and minimum (inset) intensity of eigenfunctions in the position basis of the quantum kicked rotor for $N = 32$ in the parameter range $13.8 < K < 14.8$. Shown as a continuous line is the exact density for random states while the dotted ones are the respective Gumbel and Fréchet densities.
A saddle point approximation for the orthogonal ensembles

A saddle point approximation for the inverse Laplace transform:

\[
\mathcal{L}_s^{-1}\left(\left(\pi st\right)^{m/2} e^{mnt} \text{erfc}^m\left(\sqrt{st}\right) \left(\frac{1}{\pi t}\right)^{m/2} e^{-smt} \frac{e^{-smt}}{s^{N+m/2}}\right) \approx \left(\frac{N+m}{2-2mt}\right)^{m/2}
\]
\[
\times \exp\left(\frac{m(N+m)t}{2-2mt}\right) \text{erfc}^m\left(\sqrt{(N+m)t/2-2mt}\right) \frac{1}{\Gamma\left(\frac{N+m}{2}\right)} (1-mt)^{N+m/2-1} \Theta(1-mt)
\]

gives \(F_{\text{max}}^\beta(t; N) = \)
\[
\sum_{m=0}^{k} \binom{N}{m} (-1)^m \frac{(N+m)^m}{\Gamma\left(\frac{N+m}{2}\right)} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+m}{2}\right)} (1-mt)^{N-\frac{m}{2}-1} \exp\left(\frac{m\left(\frac{N+m}{2}\right)t}{1-mt}\right) \text{erfc}^m\left(\sqrt{\frac{(N+m)\frac{t}{2}}{1-mt}}\right)
\]

or more simply using only the asymptotic form of \(\text{erfc}(z)\),

\[
F_{\text{max}}^\beta(t; N) = \sum_{m=0}^{k} \binom{N}{m} (-1)^m \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+m}{2}\right)} (1-mt)^{N+m/2-1} \frac{1}{(\pi t)^{m/2}}
\]
Comparison of the maximum intensity distribution functions for the orthogonal ensembles. The saddle point approximation improves considerably the agreement with the “exact” result from simulation vis-a-vis the asymptotic Gumbel form. The simpler form without the complementary error function is an improvement, but not good enough for small $N$ to warrant its use.
Concluding remarks

• The statistical properties of extreme intensities has not previously been applied to understanding better the eigenstates of quantum systems.

• It is possible to derive some compact, exact results for the unitary ensembles with any dimensionality and give excellent approximations for the same quantities in the orthogonal ensemble.

• The maximum intensities tend to the infinite dimensional limit very slowly and thus the functional forms contain some information about system size. They further tend toward the Gumbel distribution although, a priori, one might have expected Weibull. Means scale as $\ln a_\beta N/N$ for unitary and (roughly for) orthogonal ensembles.

• The minimum intensity statistics tend much more rapidly toward their infinite dimensional Fréchet limiting form. The mean minima are $N^{-2}$ and $\pi N^{-3}$ for the unitary and orthogonal ensembles respectively.

• These “extreme” measures give us a new way to explore non-ergodic eigenstate behaviors.

  – It would be worthwhile exploring: i) other measures such as the intensity densities in the neighborhood of the maxima or minima, and ii) how system dynamics lead to deviations from the chaotic statistical extremes.