Quantum graphs where back-scattering is prohibited

Brian Winn

School of Mathematics
Loughborough University

26th February 2008

Credits

Joint work with

Jon Harrison
and Uzy Smilansky

- *Journal of Physics A* **40** 14181–14193.

A puzzle

Find an $n \times n$ unitary matrix σ:

- with diagonal entries 0, and
- off-diagonal entries with absolute value $(n-1)^{-1/2}$.

For example (2×2)

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

HINT: $n = 3$ is impossible. . .

A puzzle

Find an $n \times n$ unitary matrix σ:

- with diagonal entries 0, and
- off-diagonal entries with absolute value $(n-1)^{-1/2}$.

For example (2×2)

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

HINT: $n = 3$ is impossible...

A puzzle

Find an $n \times n$ unitary matrix σ:

- with diagonal entries 0, and
- off-diagonal entries with absolute value $(n-1)^{-1/2}$.

For example (2×2)

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

HINT: $n = 3$ is impossible...

A puzzle

Find an $n \times n$ unitary matrix σ:

- with diagonal entries 0, and
- off-diagonal entries with absolute value $(n-1)^{-1/2}$.

For example (2×2)

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

HINT: $n = 3$ is impossible...

A puzzle

Find an $n \times n$ unitary matrix σ:

- with diagonal entries 0, and
- off-diagonal entries with absolute value $(n-1)^{-1/2}$.

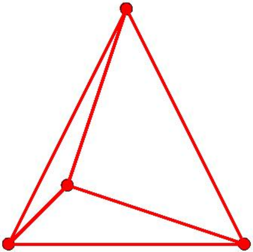
For example (2×2)

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

HINT: $n = 3$ is impossible. . .

What is a quantum graph?

A *metric graph* has bonds that have lengths $L_1, \ldots, L_\nu > 0$.



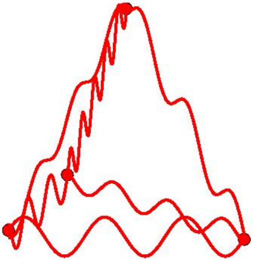
Standing waves satisfy

$$-\frac{d^2\psi_j}{dx^2} = k^2\psi_j \quad + \begin{array}{l}\text{Boundary}\\ \text{conditions}\end{array} \qquad j = 1, \ldots, \nu.$$

For $k = k_0, k_1, k_2, \ldots$ the spectrum of the quantum graph.

What is a quantum graph?

A *metric graph* has bonds that have lengths $L_1, \ldots, L_\nu > 0$.



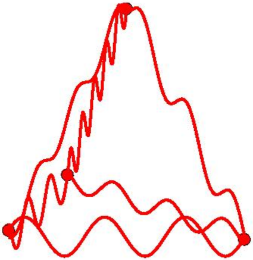
Standing waves satisfy

$$-\frac{d^2\psi_j}{dx^2} = k^2\psi_j \quad + \text{Boundary conditions} \qquad j = 1, \ldots, \nu.$$

For $k = k_0, k_1, k_2, \ldots$ the spectrum of the quantum graph.

What is a quantum graph?

A *metric graph* has bonds that have lengths $L_1, \ldots, L_\nu > 0$.

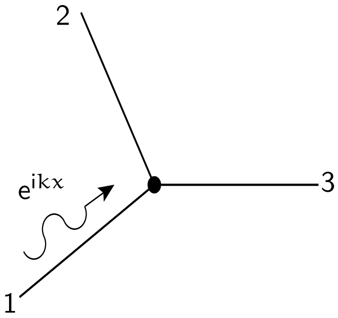


Standing waves satisfy

$$-\frac{d^2\psi_j}{dx^2} = k^2\psi_j \quad + \text{Boundary conditions} \qquad j = 1, \ldots, \nu.$$

For $k = k_0, k_1, k_2, \ldots$ the spectrum of the quantum graph.

An incoming wave is scattered at a vertex

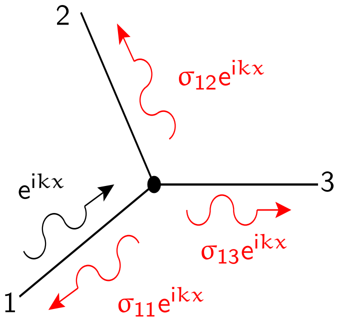


Scattering is controlled by a $d \times d$ *unitary* matrix σ. d is the degree of the vertex.

We do not say anything about the process causing the scattering.

Scattering at a vertex
(Boundary conditions for the differential equation)

An incoming wave is scattered at a vertex

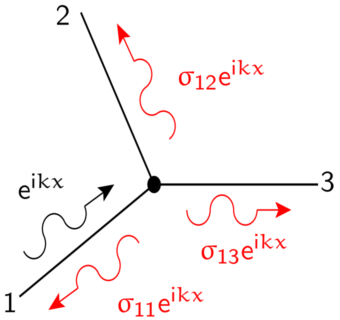


Scattering is controlled by a $d \times d$ *unitary* matrix σ. d is the degree of the vertex.

We do not say anything about the process causing the scattering.

Scattering at a vertex
(Boundary conditions for the differential equation)

An incoming wave is scattered at a vertex

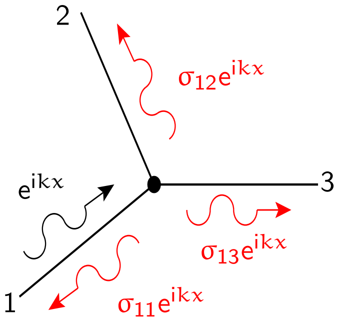


Scattering is controlled by a $d \times d$ *unitary* matrix σ. d is the degree of the vertex.

We do not say anything about the process causing the scattering.

Scattering at a vertex
(Boundary conditions for the differential equation)

An incoming wave is scattered at a vertex

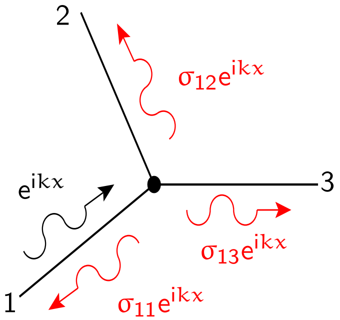


Scattering is controlled by a $d \times d$ *unitary* matrix σ. d is the degree of the vertex.

We do not say anything about the process causing the scattering.

Scattering at a vertex
(Boundary conditions for the differential equation)

An incoming wave is scattered at a vertex



Scattering is controlled by a $d \times d$ *unitary* matrix σ. d is the degree of the vertex.

We do not say anything about the process causing the scattering.

The quantum evolution operator

- Collect all entries of vertex scattering matrices σ in a $2\nu \times 2\nu$ matrix S.
- Indexing is by *directed* bonds.

In this example

$$S = \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} & \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} & \sigma_{21} & 0 & 0 \\ 0 & \sigma_{32} & \sigma_{33} & \sigma_{31} & 0 & 0 \end{pmatrix}.$$

The quantum evolution operator

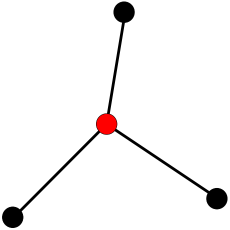
- Collect all entries of vertex scattering matrices σ in a $2\nu \times 2\nu$ matrix S.
- Indexing is by *directed* bonds.

In this example

$$S = \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} & \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} & \sigma_{21} & 0 & 0 \\ 0 & \sigma_{32} & \sigma_{33} & \sigma_{31} & 0 & 0 \end{pmatrix}.$$

The quantum evolution operator

- Collect all entries of vertex scattering matrices σ in a $2v \times 2v$ matrix S.
- Indexing is by *directed* bonds.

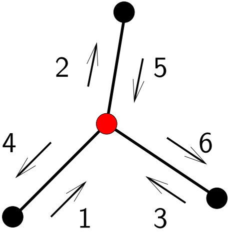


In this example

$$S = \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} & \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} & \sigma_{21} & 0 & 0 \\ 0 & \sigma_{32} & \sigma_{33} & \sigma_{31} & 0 & 0 \end{pmatrix} .$$

The quantum evolution operator

- Collect all entries of vertex scattering matrices σ in a $2v \times 2v$ matrix S.
- Indexing is by *directed* bonds.



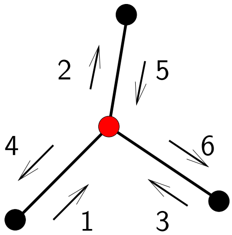
In this example

$$S = \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} & \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} & \sigma_{21} & 0 & 0 \\ 0 & \sigma_{32} & \sigma_{33} & \sigma_{31} & 0 & 0 \end{pmatrix}.$$

- Collect all entries of vertex scattering matrices σ in a $2v \times 2v$ matrix S.
- Indexing is by *directed* bonds.

Scattering matrix at centre

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}.$$



In this example

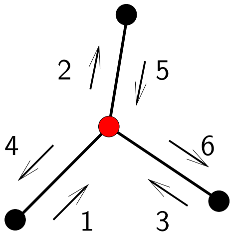
$$S = \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} & \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} & \sigma_{21} & 0 & 0 \\ 0 & \sigma_{32} & \sigma_{33} & \sigma_{31} & 0 & 0 \end{pmatrix}.$$

The quantum evolution operator

- Collect all entries of vertex scattering matrices σ in a $2v \times 2v$ matrix S.
- Indexing is by *directed* bonds.

Scattering matrix at centre

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}.$$



In this example

$$S = \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} & \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} & \sigma_{21} & 0 & 0 \\ 0 & \sigma_{32} & \sigma_{33} & \sigma_{31} & 0 & 0 \end{pmatrix}.$$

- Waves travelling along a bond of length L acquire a phase e^{ikL}.
- Put these phases into a $2\nu \times 2\nu$ diagonal matrix $D(k)$.
- Define the *quantum evolution operator* $U(k) = D(k)S$.

The spectrum

There is a standing wave of energy k^2 iff

$$\det(I - U(k)) = 0.$$

A sequence $(k_n)_{n=1}^{\infty}$ of "eigenvalues".

Alternatively: Use the von Neumann theory to construct self-adjoint extensions of the Laplace operator. . . (Kostrykin & Schrader approach.)

The quantum evolution operator
Continued

- Waves travelling along a bond of length L acquire a phase e^{ikL}.
- Put these phases into a $2\nu \times 2\nu$ diagonal matrix $D(k)$.
- Define the *quantum evolution operator* $U(k) = D(k)S$.

The spectrum

There is a standing wave of energy k^2 iff

$$\det(I - U(k)) = 0.$$

A sequence $(k_n)_{n=1}^{\infty}$ of "eigenvalues".

Alternatively: Use the von Neumann theory to construct self-adjoint extensions of the Laplace operator... (Kostrykin & Schrader approach.)

The quantum evolution operator
Continued

- Waves travelling along a bond of length L acquire a phase e^{ikL}.
- Put these phases into a $2\nu \times 2\nu$ diagonal matrix $D(k)$.
- Define the *quantum evolution operator* $U(k) = D(k)S$.

The spectrum

There is a standing wave of energy k^2 iff

$$\det(I - U(k)) = 0.$$

A sequence $(k_n)_{n=1}^{\infty}$ of "eigenvalues".

Alternatively: Use the von Neumann theory to construct self-adjoint extensions of the Laplace operator... (Kostrykin & Schrader approach.)

The quantum evolution operator
Continued

- Waves travelling along a bond of length L acquire a phase e^{ikL}.
- Put these phases into a $2v \times 2v$ diagonal matrix $D(k)$.
- Define the *quantum evolution operator* $U(k) = D(k)S$.

The spectrum

There is a standing wave of energy k^2 iff

$$\det(I - U(k)) = 0.$$

A sequence $(k_n)_{n=1}^{\infty}$ of "eigenvalues".

Alternatively: Use the von Neumann theory to construct self-adjoint extensions of the Laplace operator... (Kostrykin & Schrader approach.)

The quantum evolution operator
Continued

- Waves travelling along a bond of length L acquire a phase e^{ikL}.
- Put these phases into a $2\nu \times 2\nu$ diagonal matrix $D(k)$.
- Define the *quantum evolution operator* $U(k) = D(k)S$.

The spectrum

There is a standing wave of energy k^2 iff

$$\det(I - U(k)) = 0.$$

A sequence $(k_n)_{n=1}^{\infty}$ of "eigenvalues".

Alternatively: Use the von Neumann theory to construct self-adjoint extensions of the Laplace operator... (Kostrykin & Schrader approach.)

A classical evolution operator

- Classical dynamics is a Markov process on the directed bonds, with matrix of transition probabilities M, where

$$M_{bb'} = |U_{bb'}|^2.$$

- Since U is unitary, M is stochastic (indeed doubly stochastic).
- M has an eigenvalue 1 and all other eigenvalues have absolute value $\leqslant 1$.

A classical evolution operator

- Classical dynamics is a Markov process on the directed bonds, with matrix of transition probabilities M, where

$$M_{bb'} = |U_{bb'}|^2.$$

- Since U is unitary, M is stochastic (indeed doubly stochastic).
- M has an eigenvalue 1 and all other eigenvalues have absolute value $\leqslant 1$.

A classical evolution operator

- Classical dynamics is a Markov process on the directed bonds, with matrix of transition probabilities M, where

$$M_{bb'} = |U_{bb'}|^2.$$

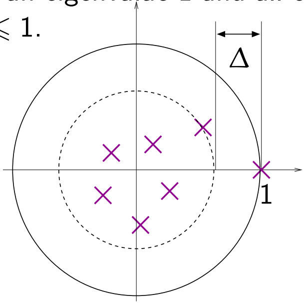
- Since U is unitary, M is stochastic (indeed doubly stochastic).
- M has an eigenvalue 1 and all other eigenvalues have absolute value $\leqslant 1$.

A classical evolution operator

- Classical dynamics is a Markov process on the directed bonds, with matrix of transition probabilities M, where

$$M_{bb'} = |U_{bb'}|^2.$$

- Since U is unitary, M is stochastic (indeed doubly stochastic).
- M has an eigenvalue 1 and all other eigenvalues have absolute value $\leqslant 1$.

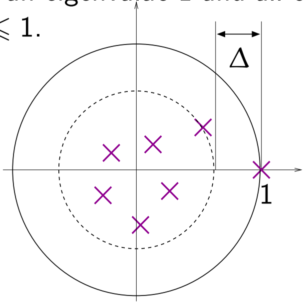


A classical evolution operator

- Classical dynamics is a Markov process on the directed bonds, with matrix of transition probabilities M, where

$$M_{bb'} = |U_{bb'}|^2.$$

- Since U is unitary, M is stochastic (indeed doubly stochastic).
- M has an eigenvalue 1 and all other eigenvalues have absolute value $\leqslant 1$.



Denote by Δ the *spectral gap*.

Random matrix conjecture

Conjecture (Tanner)

For a sequence of quantum graphs with $\nu \to \infty$, the statistics of eigenvalues converge to random matrix theory if $\nu\Delta \to \infty$.

- Gnutzmann & Altland approach. True if $\sqrt{\nu}\Delta \to \infty$. (Not a periodic orbit theory).
- Are there "nicer" choices of vertex scattering matrix?
- Quantum ergodicity? Scarring??

Random matrix conjecture

Conjecture (Tanner)

For a sequence of quantum graphs with $v \to \infty$, the statistics of eigenvalues converge to random matrix theory if $v\Delta \to \infty$.

- Gnutzmann & Altland approach. True if $\sqrt{v}\Delta \to \infty$. (Not a periodic orbit theory).
- Are there "nicer" choices of vertex scattering matrix?
- Quantum ergodicity? Scarring??

Random matrix conjecture

Conjecture (Tanner)

For a sequence of quantum graphs with $v \to \infty$, the statistics of eigenvalues converge to random matrix theory if $v\Delta \to \infty$.

- Gnutzmann & Altland approach. True if $\sqrt{v}\Delta \to \infty$. (Not a periodic orbit theory).
- Are there "nicer" choices of vertex scattering matrix?
- Quantum ergodicity? Scarring??

Random matrix conjecture

Conjecture (Tanner)

For a sequence of quantum graphs with $v \to \infty$, the statistics of eigenvalues converge to random matrix theory if $v\Delta \to \infty$.

- Gnutzmann & Altland approach. True if $\sqrt{v}\Delta \to \infty$. (Not a periodic orbit theory).
- Are there "nicer" choices of vertex scattering matrix?
- Quantum ergodicity? Scarring??

Random matrix conjecture

Conjecture (Tanner)

For a sequence of quantum graphs with $\nu \to \infty$, the statistics of eigenvalues converge to random matrix theory if $\nu\Delta \to \infty$.

- Gnutzmann & Altland approach. True if $\sqrt{\nu}\Delta \to \infty$. (Not a periodic orbit theory).
- Are there "nicer" choices of vertex scattering matrix?
- Quantum ergodicity? Scarring??

Possible vertex scattering matrices

- Neumann

$$\sigma_{jk}^{[N]} = \frac{2}{d} - \delta_{jk}.$$

Back-scattering strongly favoured.

- Fourier transform

$$\sigma_{jk}^{[F]} = \frac{1}{\sqrt{d}} e^{-2\pi i jk/d}.$$

All amplitudes equal.

Possible vertex scattering matrices

- Neumann

$$\sigma_{jk}^{[N]} = \frac{2}{d} - \delta_{jk}.$$

Back-scattering strongly favoured.

- Fourier transform

$$\sigma_{jk}^{[F]} = \frac{1}{\sqrt{d}} e^{-2\pi i jk/d}.$$

All amplitudes equal.

Possible vertex scattering matrices

- Neumann

$$\sigma_{jk}^{[N]} = \frac{2}{d} - \delta_{jk}.$$

Back-scattering strongly favoured.

- Fourier transform

$$\sigma_{jk}^{[F]} = \frac{1}{\sqrt{d}} e^{-2\pi i jk/d}.$$

All amplitudes equal.

- Equi-transmitting NEW

$$|\sigma_{jk}|^2 = \frac{1 - \delta_{jk}}{d - 1}.$$

All forward amplitudes equal; back-scattering weighted zero.

- $d = 3$, no examples (impossible).
- $d = 4$, $\sigma = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$.
- $d = 5$, $\sigma = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix}$, where $\omega = e^{2\pi i/3}$.
- Examples for d any multiple of 4, up to 184, related to existence of *skew Hadamard matrices*.
- Examples for $d = p + 1$, for all odd primes p.
- Examples for $d = 2^n$.

- $d = 3$, no examples (impossible).
- $d = 4$, $\sigma = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$.
- $d = 5$, $\sigma = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix}$, where $\omega = e^{2\pi i/3}$.
- Examples for d any multiple of 4, up to 184, related to existence of *skew Hadamard matrices*.
- Examples for $d = p + 1$, for all odd primes p.
- Examples for $d = 2^n$.

Examples

Partial solution to puzzle

- $d = 3$, no examples (impossible).
- $d = 4$, $\sigma = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$.
- $d = 5$, $\sigma = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix}$, where $\omega = e^{2\pi i/3}$.
- Examples for d any multiple of 4, up to 184, related to existence of *skew Hadamard matrices*.
- Examples for $d = p + 1$, for all odd primes p.
- Examples for $d = 2^n$.

- $d = 3$, no examples (impossible).
- $d = 4$, $\sigma = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$.
- $d = 5$, $\sigma = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix}$, where $\omega = e^{2\pi i/3}$.
- Examples for d any multiple of 4, up to 184, related to existence of *skew Hadamard matrices*.
- Examples for $d = p + 1$, for all odd primes p.
- Examples for $d = 2^n$.

- $d = 3$, no examples (impossible).
- $d = 4$, $\sigma = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$.
- $d = 5$, $\sigma = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix}$, where $\omega = e^{2\pi i/3}$.
- Examples for d any multiple of 4, up to 184, related to existence of *skew Hadamard matrices*.
- Examples for $d = p + 1$, for all odd primes p.
- Examples for $d = 2^n$.

- $d = 3$, no examples (impossible).
- $d = 4$, $\sigma = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$.
- $d = 5$, $\sigma = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix}$, where $\omega = e^{2\pi i/3}$.
- Examples for d any multiple of 4, up to 184, related to existence of *skew Hadamard matrices*.
- Examples for $d = p + 1$, for all odd primes p.
- Examples for $d = 2^n$.

The trace formula

For suitable test functions h,

$$\sum_{n=1}^{\infty} h(k_n) = \frac{\sum_b L_b}{\pi} \hat{h}(0) + \frac{1}{\pi} \sum_p \frac{A_p \ell_p}{r_p} \hat{h}(\ell_p),$$

where ℓ_p are (metric) lengths of periodic orbits, r_p is repetition number and

$$A_p = S_{b_1,b_2} S_{b_2,b_3} \cdots S_{b_n,b_1}$$

is product of elements of S accumulated on the orbit.

- With equi-transmitting scattering matrices, back-tracking orbits are eliminated.
- c.f. Ihara-Selberg zeta function.

The trace formula

For suitable test functions h,

$$\sum_{n=1}^{\infty} h(k_n) = \frac{\sum_b L_b}{\pi}\hat{h}(0) + \frac{1}{\pi}\sum_p \frac{A_p \ell_p}{r_p}\hat{h}(\ell_p),$$

where ℓ_p are (metric) lengths of periodic orbits, r_p is repetition number and

$$A_p = S_{b_1,b_2} S_{b_2,b_3} \cdots S_{b_n,b_1}$$

is product of elements of S accumulated on the orbit.

- With equi-transmitting scattering matrices, back-tracking orbits are eliminated.
- c.f. Ihara-Selberg zeta function.

The trace formula

For suitable test functions h,

$$\sum_{n=1}^{\infty} h(k_n) = \frac{\sum_b L_b}{\pi}\hat{h}(0) + \frac{1}{\pi}\sum_p \frac{A_p \ell_p}{r_p}\hat{h}(\ell_p),$$

where ℓ_p are (metric) lengths of periodic orbits, r_p is repetition number and

$$A_p = S_{b_1,b_2} S_{b_2,b_3} \cdots S_{b_n,b_1}$$

is product of elements of S accumulated on the orbit.

- With equi-transmitting scattering matrices, back-tracking orbits are eliminated.
- c.f. Ihara-Selberg zeta function.

The trace formula

For suitable test functions h,

$$\sum_{n=1}^{\infty} h(k_n) = \frac{\sum_b L_b}{\pi}\hat{h}(0) + \frac{1}{\pi}\sum_p \frac{A_p \ell_p}{r_p}\hat{h}(\ell_p),$$

where ℓ_p are (metric) lengths of periodic orbits, r_p is repetition number and

$$A_p = S_{b_1,b_2} S_{b_2,b_3} \cdots S_{b_n,b_1}$$

is product of elements of S accumulated on the orbit.

- With equi-transmitting scattering matrices, back-tracking orbits are eliminated.
- c.f. Ihara-Selberg zeta function.

The trace formula

For suitable test functions h,

$$\sum_{n=1}^{\infty} h(k_n) = \frac{\sum_b L_b}{\pi} \hat{h}(0) + \frac{1}{\pi} \sum_p \frac{A_p \ell_p}{r_p} \hat{h}(\ell_p),$$

where ℓ_p are (metric) lengths of periodic orbits, r_p is repetition number and

$$A_p = S_{b_1,b_2} S_{b_2,b_3} \cdots S_{b_n,b_1}$$

is product of elements of S accumulated on the orbit.

- With equi-transmitting scattering matrices, back-tracking orbits are eliminated.
- c.f. Ihara-Selberg zeta function.

Further Ihara connections

In a regular graph all vertices have the same degree.

Theorem 1 (Harrison, Smilansky, W.)

For a regular quantum graph with equi-transmitting vertex scattering matrices the eigenvalues of $\mathcal{M}$ are (up to scaling) at the positions of the poles of the Ihara-Selberg zeta function.

This allows us to prove:

Theorem 2 (Harrison, Smilansky, W.)

For a regular quantum graph with equi-transmitting vertex scattering matrices the spectral gap is strictly greater than the spectral gap for the same graph with Neumann or Fourier transform scattering matrices.

Further Ihara connections

In a regular graph all vertices have the same degree.

Theorem 1 (Harrison, Smilansky, W.)

For a regular quantum graph with equi-transmitting vertex scattering matrices the eigenvalues of $\mathcal{M}$ are (up to scaling) at the positions of the poles of the Ihara-Selberg zeta function.

This allows us to prove:

Theorem 2 (Harrison, Smilansky, W.)

For a regular quantum graph with equi-transmitting vertex scattering matrices the spectral gap is strictly greater than the spectral gap for the same graph with Neumann or Fourier transform scattering matrices.

Further Ihara connections

In a regular graph all vertices have the same degree.

Theorem 1 (Harrison, Smilansky, W.)

For a regular quantum graph with equi-transmitting vertex scattering matrices the eigenvalues of $\mathcal{M}$ *are (up to scaling) at the positions of the poles of the Ihara-Selberg zeta function.*

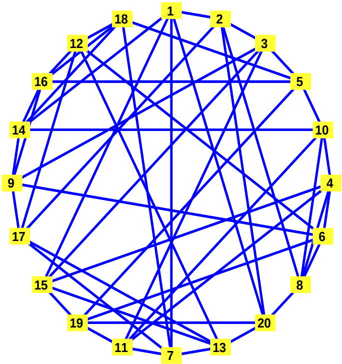
This allows us to prove:

Theorem 2 (Harrison, Smilansky, W.)

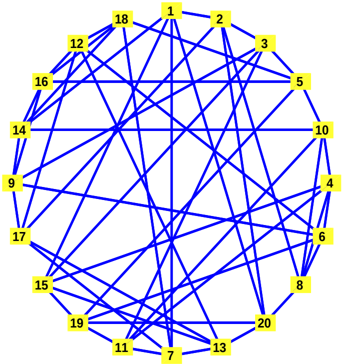
For a regular quantum graph with equi-transmitting vertex scattering matrices the spectral gap is strictly greater than the spectral gap for the same graph with Neumann or Fourier transform scattering matrices.

Spectral statistics for an example

We considered the 5-regular graph



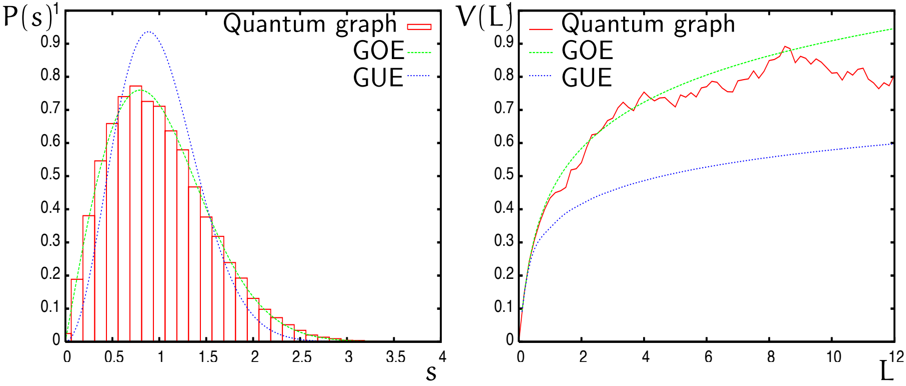
We considered the 5-regular graph



with vertex scattering matrix

$$\sigma = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & \omega^2 & \omega & 1 & 0 \end{pmatrix}$$

at each vertex.



Nearest neighbour spacing

Number variance

Lies
To simplify the exposition

- Specifying vertex scattering matrix is not equivalent to self-adjoint extension of the Laplace operator.
- I omitted a technical condition from Theorem 2.
- I did not compute eigenvalues to draw the figures—instead I averaged statistics of eigenphases of $U(k)$ over bond lengths.

There are no lies in the article.

Lies

To simplify the exposition

- Specifying vertex scattering matrix is not equivalent to self-adjoint extension of the Laplace operator.
- I omitted a technical condition from Theorem 2.
- I did not compute eigenvalues to draw the figures—instead I averaged statistics of eigenphases of $U(k)$ over bond lengths.

There are no lies in the article.

- Specifying vertex scattering matrix is not equivalent to self-adjoint extension of the Laplace operator.
- I omitted a technical condition from Theorem 2.
- I did not compute eigenvalues to draw the figures—instead I averaged statistics of eigenphases of $U(k)$ over bond lengths.

There are no lies in the article.

Lies

To simplify the exposition

- Specifying vertex scattering matrix is not equivalent to self-adjoint extension of the Laplace operator.
- I omitted a technical condition from Theorem 2.
- I did not compute eigenvalues to draw the figures—instead I averaged statistics of eigenphases of $U(k)$ over bond lengths.

There are no lies in the article.

Outlook

- Equi-transmitting matrices of other dimensions. First open dimension is $d = 7$.
- Can one "hear" the shape of quantum graphs with equi-transmitting scattering matrices? Gutkin & Smilansky proof breaks down here.
- Are there other interesting connections to discrete graph objects?

Outlook

- Equi-transmitting matrices of other dimensions. First open dimension is $d = 7$.
- Can one "hear" the shape of quantum graphs with equi-transmitting scattering matrices? Gutkin & Smilansky proof breaks down here.
- Are there other interesting connections to discrete graph objects?

Outlook

- Equi-transmitting matrices of other dimensions. First open dimension is $d = 7$.
- Can one "hear" the shape of quantum graphs with equi-transmitting scattering matrices? Gutkin & Smilansky proof breaks down here.
- Are there other interesting connections to discrete graph objects?