On the Hubbard-Stratonovich Transformation for Interacting Bosons

Martin R. Zirnbauer

Banff (February 28, 2008)

- Hubbard-Stratonovich for fermions: a reminder
- Bosons are different !
- Random matrices: hyperbolic HS transformation made rigorous
- Consequences for interacting bosons

Weyl group symmetry

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a compact Lie group G on a vector space V.

The character $\chi(g) := \mathrm{Tr}_V\, \rho(g)$ is a radial function:

$$\chi(h) = \chi(g\, h\, g^{-1}) \,, \qquad g\,,\, h \in G.$$

Therefore the restriction to a maximal torus $T \subset G$ is invariant under the action of the Weyl group $W = N_G(T)/Z_G(T)$:

$$\chi(t) = \chi(w(t)) \,, \qquad w(t) = g_w\, t\, g_w^{-1}$$

Weyl character formula

For any irreducible representation one has

highest weight

$$\chi(t) = \mathrm{Sym}_W\left(\frac{\mathrm{e}^{\lambda(\ln t)}}{\prod_{\alpha>0}(1-\mathrm{e}^{-\alpha(\ln t)})}\right) .$$

Weyl group

positive roots

Example : $G = \mathrm{SU}(2)$, $V = \mathbb{C}^{2S+1}$:

$$\chi(\mathrm{e}^{\mathrm{i}\theta\sigma_3/2}) = \sum_{m=-S}^{+S} \mathrm{e}^{\mathrm{i}m\theta} = \frac{\sin((S+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)} = \frac{\mathrm{e}^{\mathrm{i}S\theta}}{1-\mathrm{e}^{-\mathrm{i}\theta}} + \frac{\mathrm{e}^{-\mathrm{i}S\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}} .$$

Relevance for discussions here

The CUE generating function ($\mathrm{Im}\,\varepsilon_{A,B} > 0$)

$$Z(\varepsilon_A, \varepsilon_B, \varepsilon_C, \varepsilon_D) = \left\langle \frac{\mathrm{Det}(1-\mathrm{e}^{\mathrm{i}\varepsilon_C}U)\ \mathrm{Det}(1-\mathrm{e}^{-\mathrm{i}\varepsilon_D}U)}{\mathrm{Det}(1-\mathrm{e}^{\mathrm{i}\varepsilon_A}U)\ \mathrm{Det}(1-\mathrm{e}^{-\mathrm{i}\varepsilon_B}U)} \right\rangle_{\mathrm{CUE}}$$

is the character of an irreducible representation of $\mathfrak{gl}(2|2)$ (on the subspace of $\mathrm{U}(N)$ – invariant states of a Fock space of $\mathrm{U}(N)$ – fundamental bosons and fermions).

Conrey, Farmer, MRZ; Huckleberry, Püttmann, MRZ: The Weyl character formula applies in this setting!

$$Z(\varepsilon_A, \varepsilon_B, \varepsilon_C, \varepsilon_D) = \mathrm{Sym}_W \left(\mathrm{e}^{\mathrm{i}N(\varepsilon_B - \varepsilon_D)} \frac{r(\varepsilon_A + \varepsilon_D)\ r(\varepsilon_B + \varepsilon_C)}{r(\varepsilon_A + \varepsilon_B)\ r(\varepsilon_C + \varepsilon_D)} \right),$$

$$r(x) = 1 - \mathrm{e}^{-\mathrm{i}x}, \quad W = \mathbb{Z}_2 = \{1, w\}, \quad w:\ (\varepsilon_C, \varepsilon_D) \mapsto (-\varepsilon_D, -\varepsilon_C).$$

Hubbard-Stratonovich Method for Fermions

Basic Setting

Matrix elements of the evolution operator :

$$S(t_f, t_i) = \left\langle \psi_f \,|\, \mathrm{e}^{-\mathrm{i}(t_f - t_i)H} \,|\, \psi_i \right\rangle = \left\langle \psi_f \,|\, \mathrm{e}^{-\mathrm{i}\varepsilon H} \mathrm{e}^{-\mathrm{i}\varepsilon H} \times \cdots \times \mathrm{e}^{-\mathrm{i}\varepsilon H} \,|\, \psi_i \right\rangle$$

N factors, $t_f - t_i = N\varepsilon$.

Hamiltonian (kinetic energy plus two – body interaction) :

$$H = T + V, \quad T = \sum_{\alpha\beta} t_{\alpha\beta}\, c_\alpha^* c_\beta \ , \quad V = \tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} v_{\alpha\beta,\gamma\delta}\, c_\alpha^* c_\beta^* c_\delta c_\gamma \ .$$

To set up the functional integral one factors

$$\mathrm{e}^{-\mathrm{i}\varepsilon (T+V)} \cong \mathrm{e}^{-\mathrm{i}\varepsilon T} \mathrm{e}^{-\mathrm{i}\varepsilon V} \quad (\varepsilon \to 0).$$

Hubbard-Stratonovich Decoupling

Express two – body interaction by density operators :

$$V = \tfrac{1}{2}\sum_{mn} W_{m,n}\,\rho_m\,\rho_n\ , \quad W_{\alpha\gamma,\beta\delta} = V_{\alpha\beta,\gamma\delta}\ , \qquad \rho_{\alpha\gamma} = c_\alpha^*\,c_\gamma\ .$$

Decouple using auxiliary field : $\mathrm{e}^{-\mathrm{i}\varepsilon V} = \mathrm{e}^{-(\mathrm{i}\varepsilon/2)\sum_{mn} W_{m,n}\,\rho_m\,\rho_n}$

$$= \mathrm{Det}^{-1/2}(2\pi\,\mathrm{i}W/\varepsilon)\int \mathrm{D}\sigma\ \mathrm{e}^{(\mathrm{i}\varepsilon/2)\sum_{mn}(W^{-1})_{m,n}\,\sigma_m\,\sigma_n - \mathrm{i}\varepsilon\sum_n \sigma_n\,\rho_n}\,.$$

Final result (schematic) :

$$S(t_f\,,\,t_i) = \int \mathfrak{D}\sigma\ \mathrm{e}^{\mathrm{i}\int_{t_i}^{t_f} dt\ \sigma(t)\,W^{-1}\sigma(t)}\left\langle \psi_f\,|\,\mathfrak{T}\,\mathrm{e}^{-\mathrm{i}\int_{t_i}^{t_f} dt\ (T+\sigma(t))\,\rho}\ |\,\psi_i\right\rangle$$

Stationary – phase is equivalent to Hartree – Fock approximation.

Fluctuations of Gaussian order yield RPA corrections.

Fermions: a variant of the method

Write the partition function as a functional integral over Grassmann fields:

$$\mathrm{Tr}\,\mathrm{e}^{-\beta H} = \int \mathfrak{D}\psi\ \mathrm{e}^{\int_0^\beta dt\,\langle \psi | \partial_t - H | \psi \rangle} .$$

The expectation $\langle \psi | H | \psi \rangle$ is obtained by replacing

$$c_\alpha^* \to \overline{\psi}_\alpha\ ,\quad c_\alpha \to \psi_\alpha\ .$$

Hubbard – Stratonovich decoupling (as before):

$$\mathrm{e}^{-(\varepsilon/2)\sum_{mn} W_{m,n}\rho_m\rho_n} = \int \mathrm{D}\sigma\ \mathrm{e}^{-(\varepsilon/2)\sum_{mn}(W^{-1})_{m,n}\sigma_m\sigma_n + \mathrm{i}\varepsilon\sum_n \rho_n\sigma_n}$$

where $\rho_{\alpha\gamma} = \overline{\psi}_\alpha\psi_\gamma$.

A model for bosons (prototype)

Charge $Q = \sum_{j=1}^{N} a_j^* a_j$, boson pair $P = \sum_{j=1}^{N} a_j a_j$,

Hamiltonian $H = \nu Q^2 + \lambda P^* P$, $|\lambda| < \nu$ (for stability) .

Use the method of boson – coherent states :

$$a_j \to z_j \ , \ \ a_j^* \to \bar{z}_j \ , \ \ H \to \nu \Big(\sum \bar{z}_j z_j \Big)^2 + \lambda \Big(\sum \bar{z}_j \bar{z}_j \Big) \Big(\sum z_j z_j \Big) .$$

Notation : $B(X) = \nu X_0^2 + \lambda (X_1^2 + X_2^2)$,

$$X_0 = \sum \bar{z}_j z_j \ , \quad X_1 = \mathrm{Re} \sum z_j z_j \ , \quad X_2 = \mathrm{Im} \sum z_j z_j \ .$$

Where's the difficulty?

Let the interaction be of mixed type: $\nu > 0$ (repulsive), $\lambda < 0$ (attractive). Then the quadratic form

$$B(X) = \nu X_0^2 + \lambda (X_1^2 + X_2^2)$$

is of indefinite signature (1,2). Hubbard – Stratonovich (?):

$$\mathrm{e}^{-\varepsilon B(X)} = C_{\varepsilon,B} \int_D \mathrm{e}^{-\varepsilon B^{-1}(Y) + \mathrm{i}\varepsilon \langle X,Y \rangle} d^3 Y \,.$$

All of the desired properties for the integration domain,

(i) $D \subset \mathbb{R}^3$, (ii) $B^{-1}|_D > 0$, (iii) $\partial D = 0$,

cannot be met.

Hyperbolic Hubbard-Stratonovich Transformation Made Rigorous

- Y. Fyodorov, Y. Wei, MRZ, arxiv:0801.4960
- Fyodorov, J. Phys. Condensed Matter **17** (2005) S1915
- Wei & Fyodorov, J. Phys. A **40** (2007) 13587

Background & motivation

Consider real symmetric $N \times N$ matrices H

$$\left\langle \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}\, HK} \right\rangle_{\mathrm{GOE}} = \int \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}\, HK} d\mu(H) = \mathrm{e}^{-(b^2/2N)\,\mathrm{Tr}\, K^2}$$

Expectation of reciprocal of product of characteristic polynomials:

$$F(z) = \left\langle \prod_{j=1}^{p+q} \mathrm{Det}^{-1/2}(z_j - H) \right\rangle_{\mathrm{GOE}} \qquad (z_j \in \mathbb{C} \setminus \mathbb{R})$$

with $\mathrm{Im}\, z_j > 0$ for $j = 1, \ldots, p$ and $\mathrm{Im}\, z_j < 0$ for $j = p+1, \ldots, p+q$

Use $\mathrm{Det}^{-1/2}(z_j - H) = (\mathrm{i}\pi s_j)^{-N/2} \int \exp \mathrm{i}\, s_j (\varphi, \varphi z_j - H\varphi)\, |d\varphi|$,

to obtain $\qquad s_j = \mathrm{sign}\, \mathrm{Im}(z_j)$

$$F(z) = \int \mathrm{e}^{\mathrm{i} \sum_{j=1}^{p+q} s_j z_j (\varphi_j, \varphi_j) - (b^2/2N)\mathrm{Tr}\, A(\varphi)^2} \prod_j (\mathrm{i}\pi s_j)^{-N/2} |d\varphi_j|$$

Background & motivation (continued)

$$A(\varphi)_{ij} = \sum_{a=1}^{N} \varphi_{i,a}\, \varphi_{j,a}\, s_j$$

Note : $A^{\mathrm{t}} = sAs$ with $s = \mathrm{diag}(\mathrm{Id}_p\,,\, -\mathrm{Id}_q)$ is invariant under conjugation by $g^{-1} = sg^{\mathrm{t}}s \in \mathrm{O}_{p,q}$ ("hyperbolic symmetry", Wegner 1979)

Next step is Hubbard-Stratonovich decoupling:

$$C_0\ \mathrm{e}^{-\mathrm{Tr}\,A^2} = \int_D \mathrm{e}^{-\mathrm{Tr}\,R^2 - 2\mathrm{i}\,\mathrm{Tr}\,AR}\, |\, dR\,|$$

Problem: how is this done correctly?!

$A^{\mathrm{t}} = sAs$ forces $R^{\mathrm{t}} = sRs$ but $\mathrm{Tr}\,R^2 = \mathrm{Tr}\,RsR^{\mathrm{t}}s$ is of indefinite sign.

Domain of Schäfer – Wegner (1980) is not $\mathrm{O}_{p,q}$ – invariant.

Pruisken-Schäfer domain

$Rv = \lambda v$:

$v^{\mathrm{t}} s v > 0$ space-like, $\bullet$

$v^{\mathrm{t}} s v < 0$ time-like, $\circ$

Every $\mathrm{O}_{p,q}$ – diagonalizable matrix R has p space – like and q time – like eigenvalues.

Encode ordering by motif, e.g., $\sigma(R) = \bullet\circ\bullet\circ\circ\bullet \quad (p = q = 3)$.
Associate with each motif σ a domain D_σ by closure.

Pruisken – Schäfer domain $D = \bigcup_\sigma D_\sigma$ is a union of $\binom{p+q}{p} = \binom{p+q}{q}$ domains. Each D_σ is $\mathrm{O}_{p,q}$ – invariant.
$D_\sigma \cap D_{\sigma'}$ for $\sigma \neq \sigma'$ has co – dimension 2.

Statement of result

Let $|dR| = \prod_{i\le j} dR_{ij}$ (Lebesgue measure)

Theorem (FWZ). There exists some choice of cutoff function $R \mapsto \chi_\varepsilon(R)$ (converging pointwise to unity as $\varepsilon \to 0$) and a unique choice of sign function $\sigma \mapsto \mathrm{sgn}(\sigma) \in \{\pm 1\}$ and a number $C_{p,q}$ such that

$$C_{p,q} \lim_{\varepsilon\to 0} \sum_\sigma \mathrm{sgn}(\sigma) \int_{D_\sigma} \mathrm{e}^{-\mathrm{Tr}\, R^2 - 2\mathrm{i}\,\mathrm{Tr}\, AR} \chi_\varepsilon(R)\, |dR| \; = \; \mathrm{e}^{-\mathrm{Tr}\, A^2}$$

holds true for all matrices $A = sA^{\mathrm{t}}s$ with the property $As > 0$.

Remark. $\mathrm{sgn}(\sigma)$ is the parity of the number of transpositions $\bullet \leftrightarrow \circ$ needed to reduce σ to the extremal form $\sigma_0 = \bullet\bullet\ldots\bullet\circ\circ\ldots\circ$.
The proof is *not* by completing the square and shifting.

Idea of proof

Remark. $p = q = 1$: Fyodorov 2005

$p = 2,\ q = 1$: Wei, Fyodorov 2007

Consider

$$I(A) := \lim_{\varepsilon \to 0} \sum_\sigma \operatorname{sgn}(\sigma) \int_{D_\sigma} \mathrm{e}^{-\operatorname{Tr}(R+\mathrm{i}A)^2} \chi_\varepsilon(R+\mathrm{i}A)\,|\,dR\,|\,.$$

Show that

$$\frac{d}{dt} I(A + t\dot{A})\bigg|_{t=0} = \lim_{\varepsilon \to 0} \mathrm{i} \sum_\sigma \operatorname{sgn}(\sigma) \int_{\partial D_\sigma} \mathrm{e}^{-\operatorname{Tr}(R+\mathrm{i}A)^2} \chi_\varepsilon(R+\mathrm{i}A)\ \iota_{\tau(\dot{A})}\,|\,dR\,|$$

vanishes.

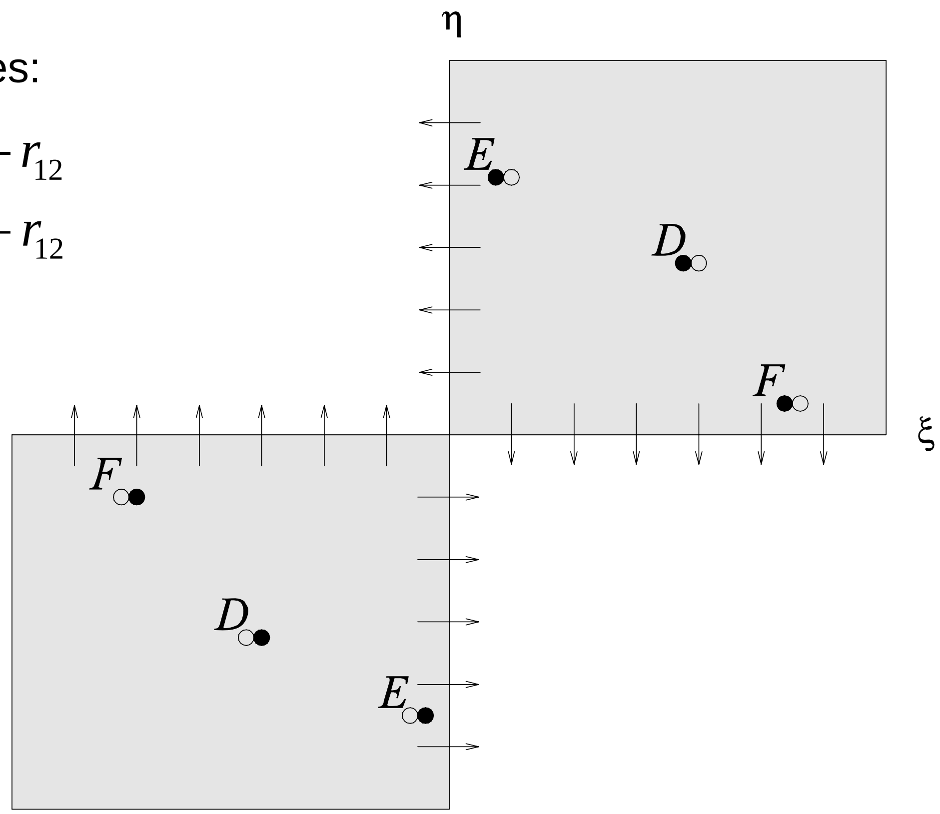
Two domains for p = q = 1

$$R = \begin{pmatrix} r_{11} & r_{12} \\ -r_{12} & r_{22} \end{pmatrix}$$

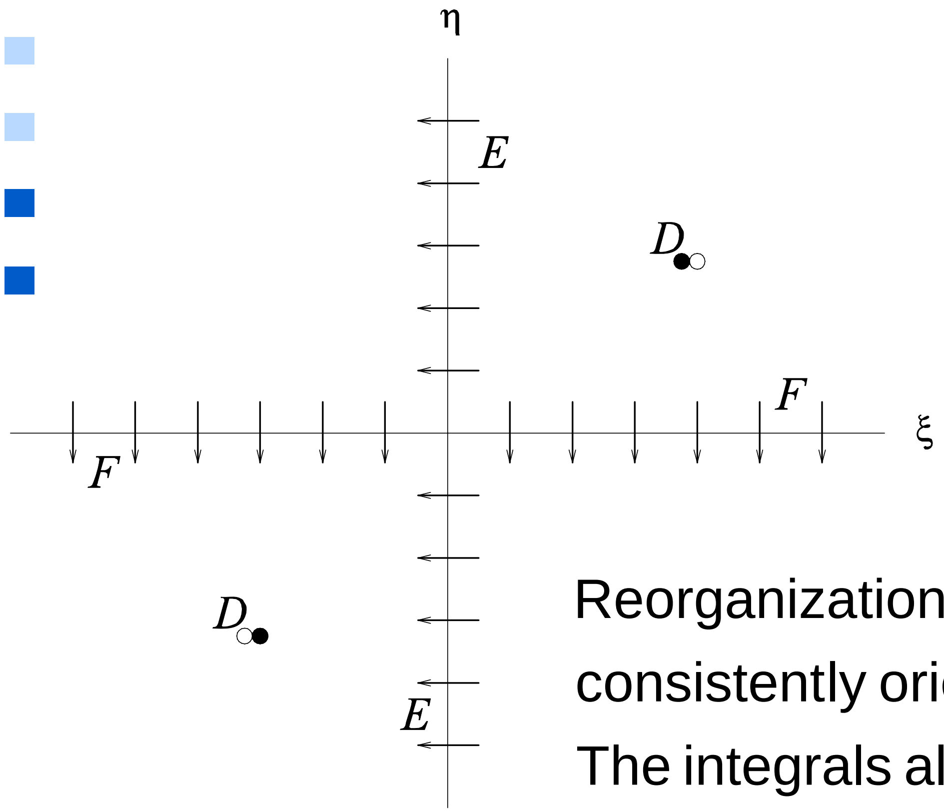
Light-cone coordinates:

$$\xi = (r_{11} - r_{22})/2 - r_{12}$$

$$\eta = (r_{11} - r_{22})/2 + r_{12}$$



Reorganization of boundary components



Reorganization of the boundary pieces gives two consistently oriented boundary planes E and F. The integrals along E and F are exponential and oscillatory and vanish in the limit $\varepsilon \to 0$.

Corollary

Formulation in terms of eigenvalues: Let $R = g\lambda g^{-1}$ with $\lambda = \mathrm{diag}(\lambda_1, ..., \lambda_{p+q})$ and $g \in \mathrm{SO}_{p,q}$. Volume element $|dR| = J(\lambda)\,|d\lambda|\,dg$ where $|d\lambda| = \prod_{i=1}^{p+q} d\lambda_i$ and dg Haar measure for $\mathrm{SO}_{p,q}$. Jacobian: $J(\lambda) = \prod_{i<j} |\lambda_i - \lambda_j|$.

Corollary (FWZ). Define a sign-alternating Jacobian $J'(\lambda)$ by

$$J'(\lambda) = J(\lambda) \prod_{i=1}^{p} \prod_{j=p+1}^{p+q} \mathrm{sign}(\lambda_i - \lambda_j) = J(\lambda)\,\mathrm{sgn}(\sigma(\lambda)).$$

Then, if A satisfies the condition $sA^{\mathrm{t}} = As > 0$ we have

$$C_{p,q} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{p+q}} \left(\int_{\mathrm{SO}_{p,q}} \mathrm{e}^{-2\mathrm{i}\,\mathrm{Tr}\, A g \lambda g^{-1}} \chi_\varepsilon(g\lambda g^{-1})\, dg \right) \mathrm{e}^{-\mathrm{Tr}\,\lambda^2} J'(\lambda)\,|d\lambda| \;=\; \mathrm{e}^{-\mathrm{Tr}\, A^2}.$$