On pie and noodles

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Department of Mathematics

Aggieland Saturday, 18 Feb 2012

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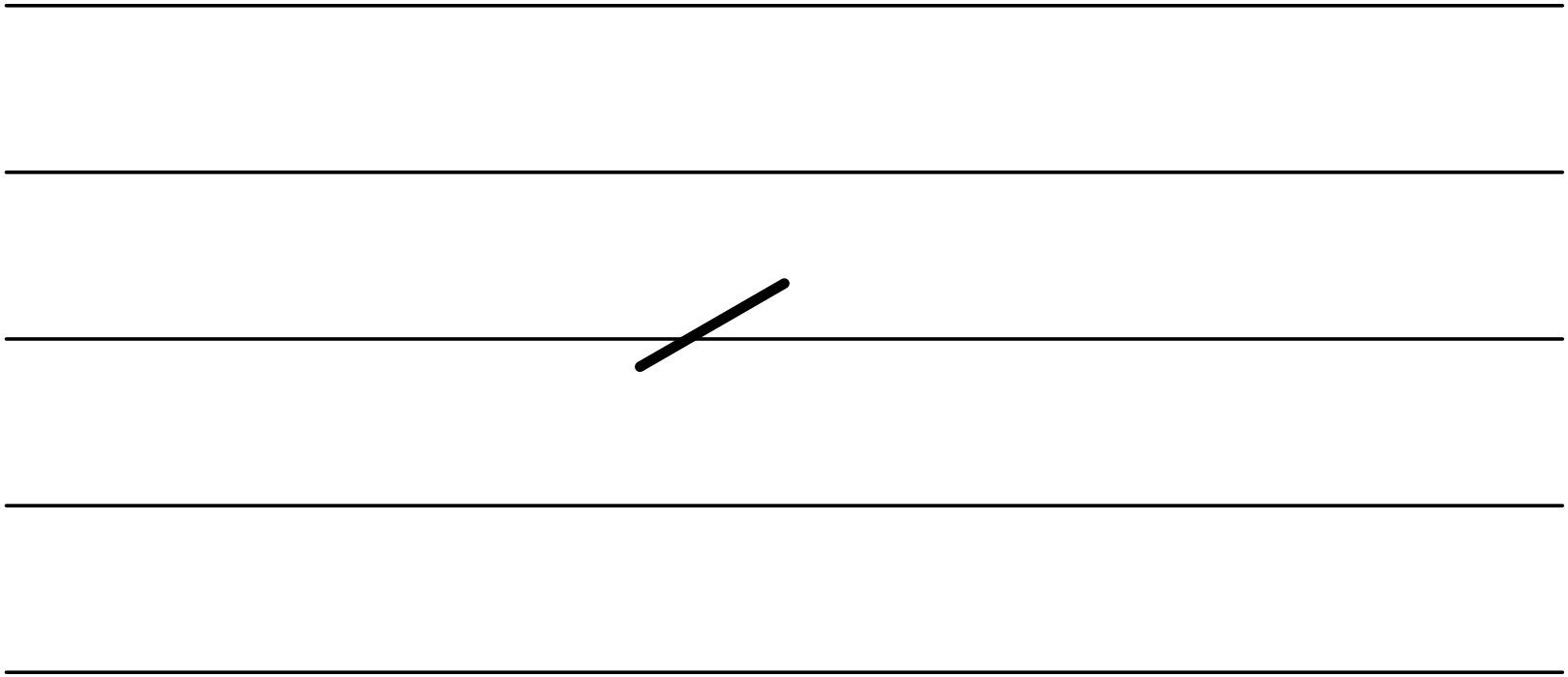

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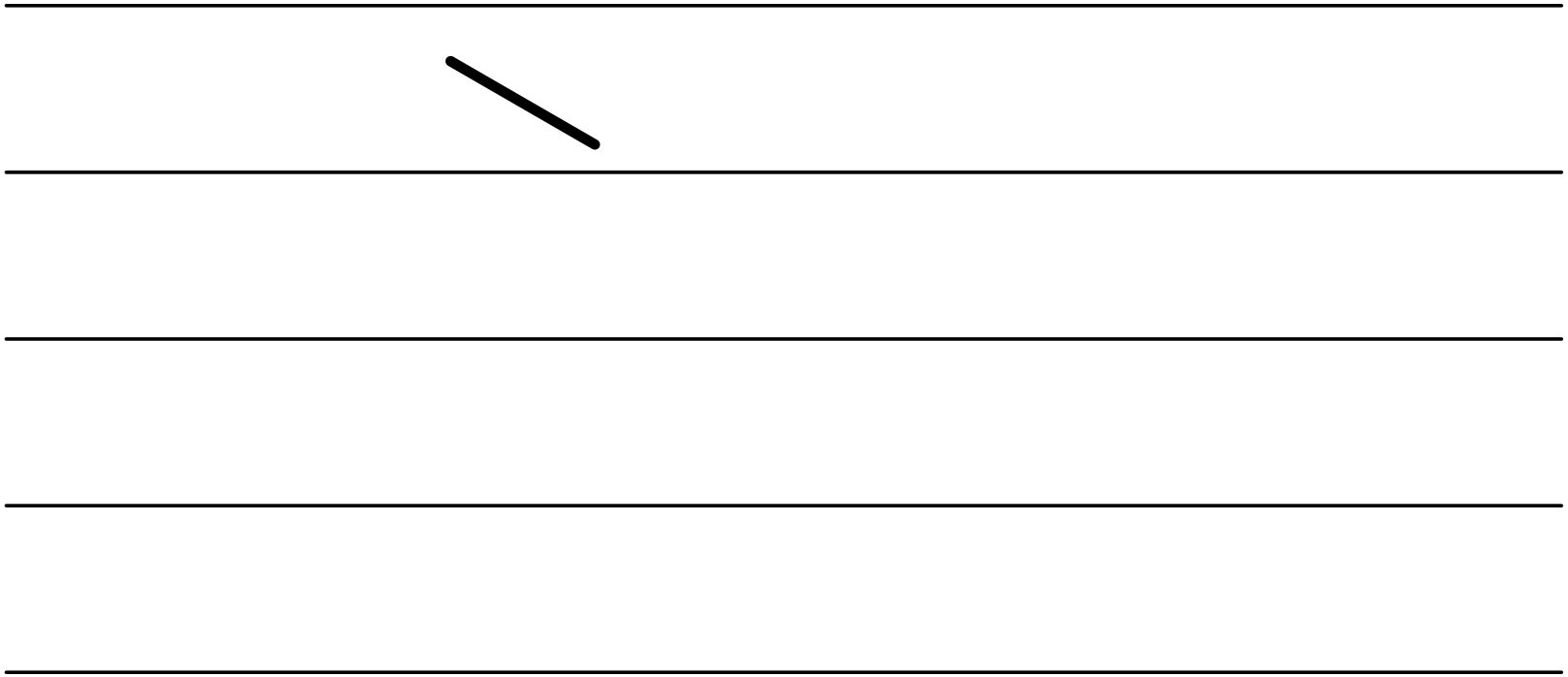

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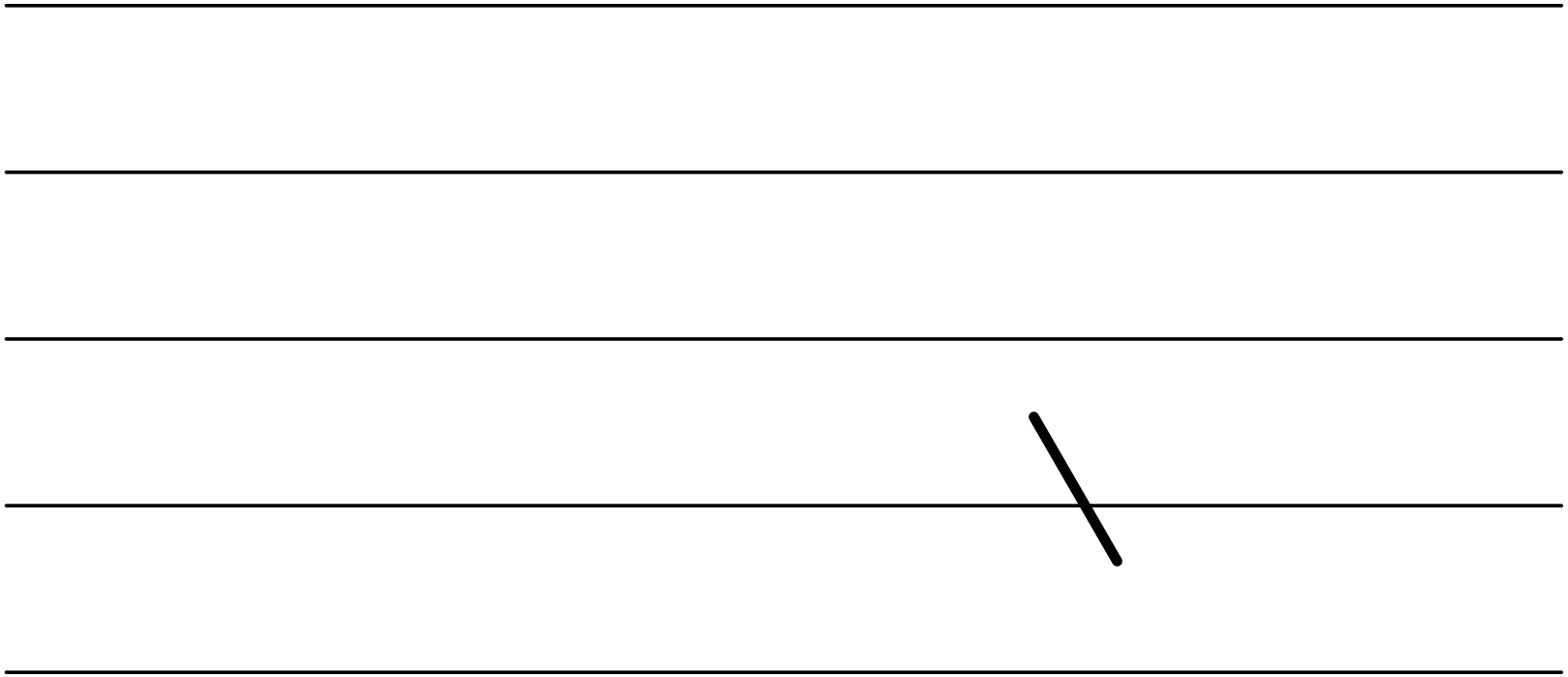

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- He was not "Buffon" when he invented "Buffon's needle".

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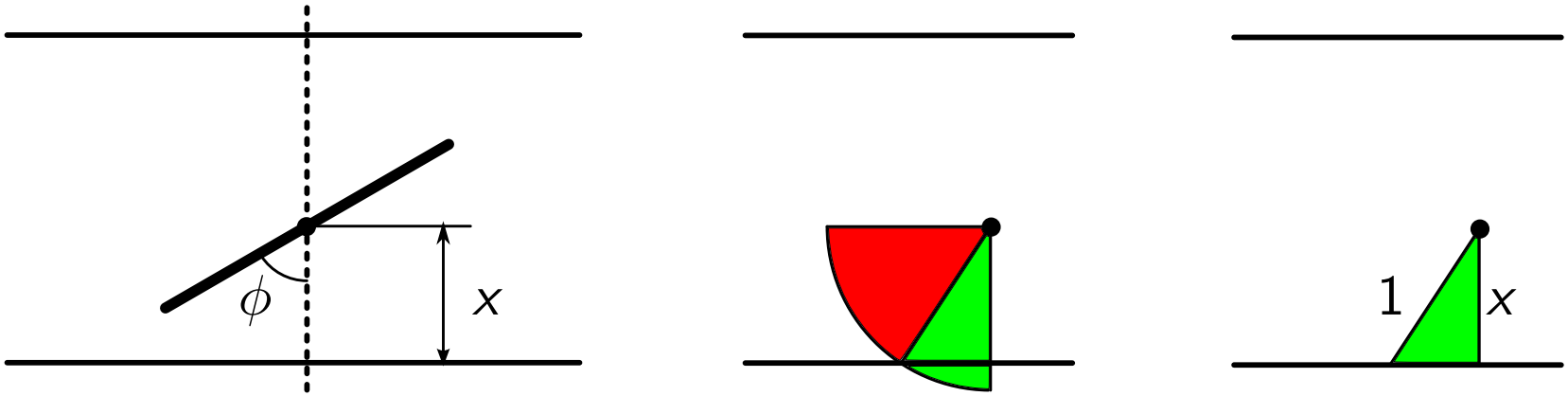

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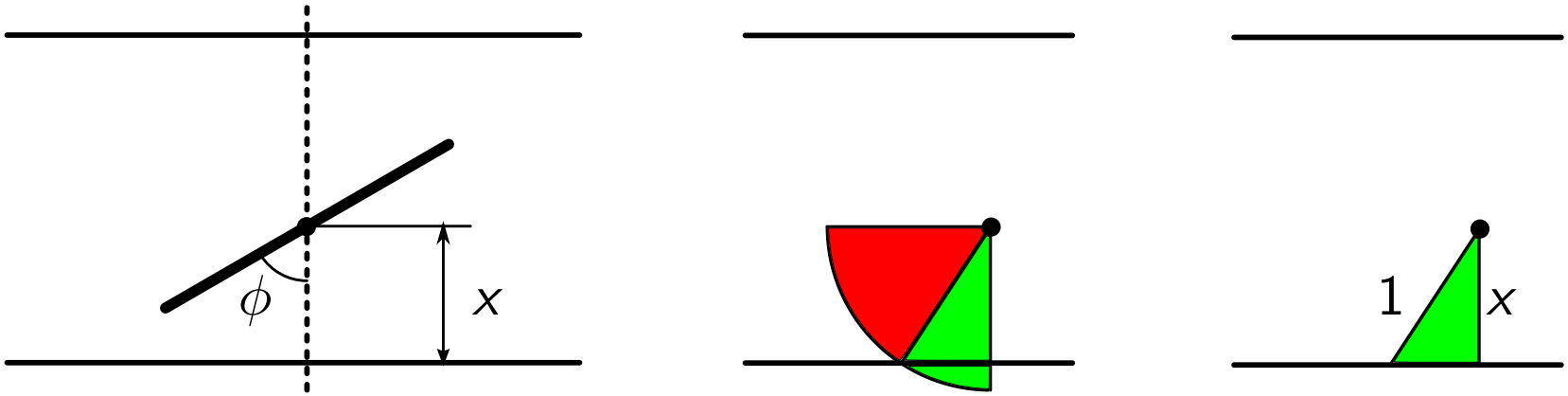

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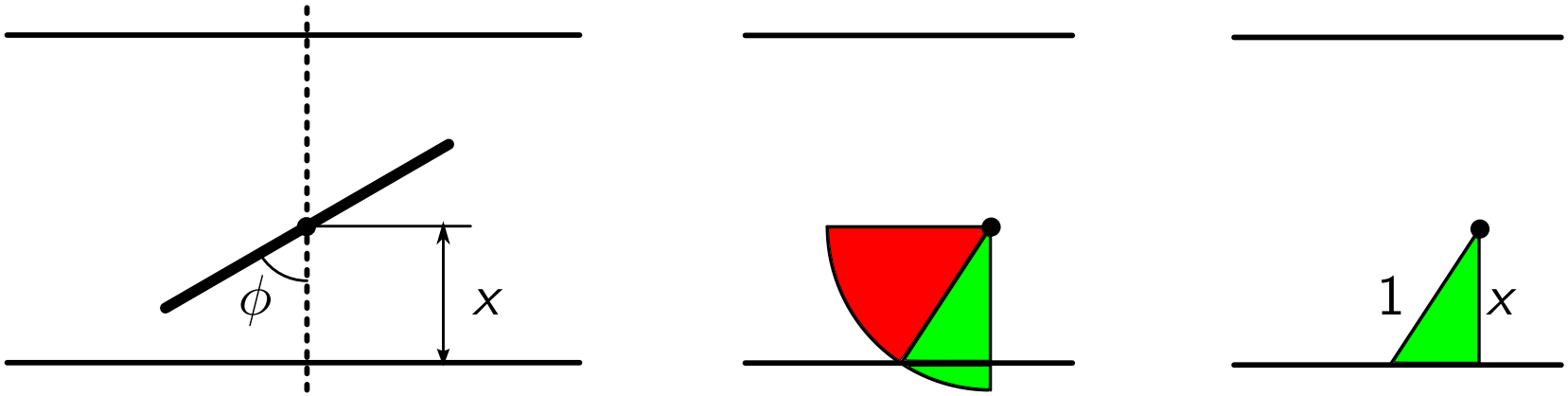

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Probability is $\frac{\pi}{2}$

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- The constant π is related to things round. But everything in this problem is straight! What is going on?

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  - Which is just $-\frac{2}{3} + \frac{1}{2}$!

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- This is a general mathematical law.

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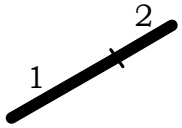
- Instead of finding probability we can look for the expectation.

Bending the needle

- Put a mark on the needle that separates the needle into two parts, part 1 and part 2.



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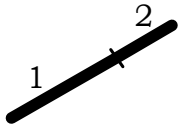

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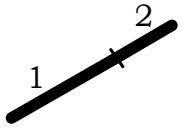

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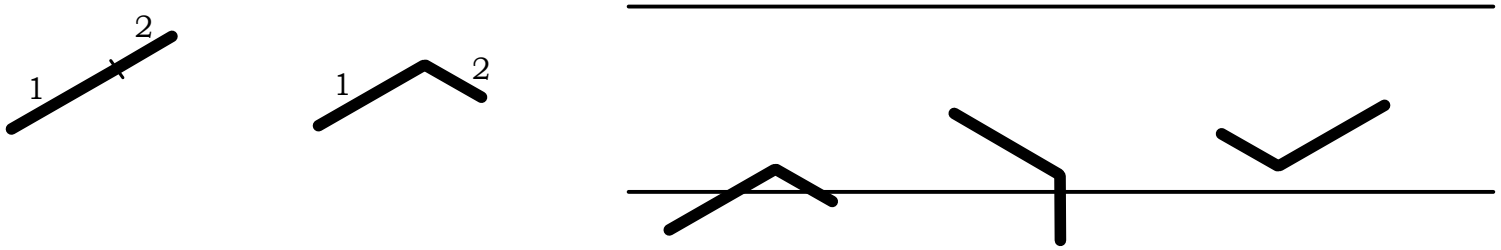

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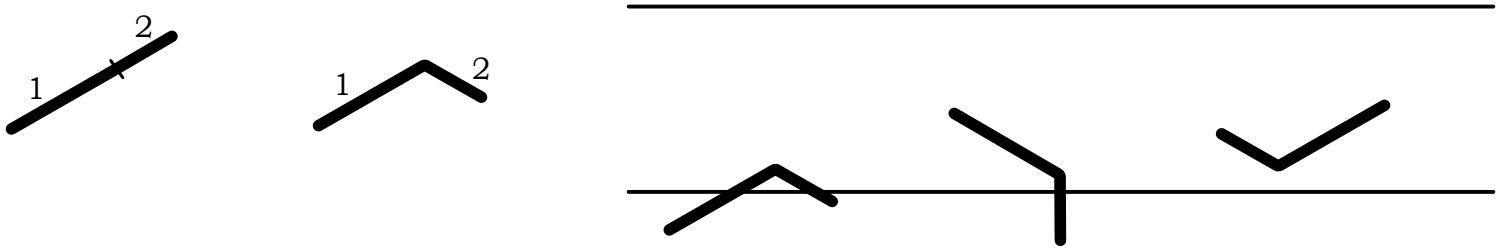

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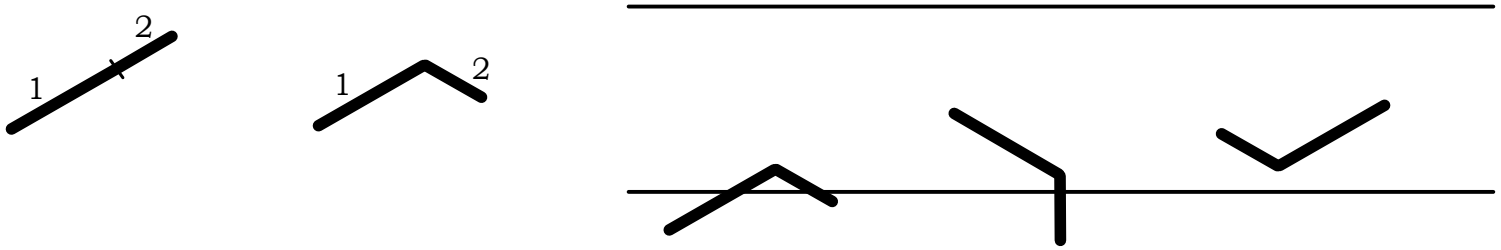

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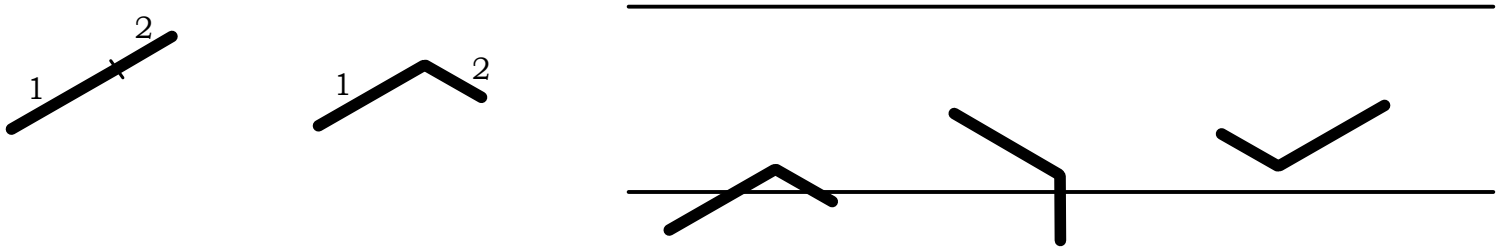

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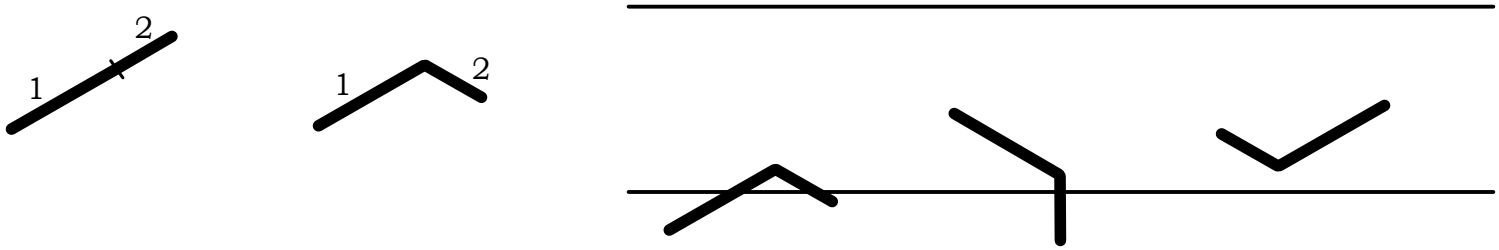

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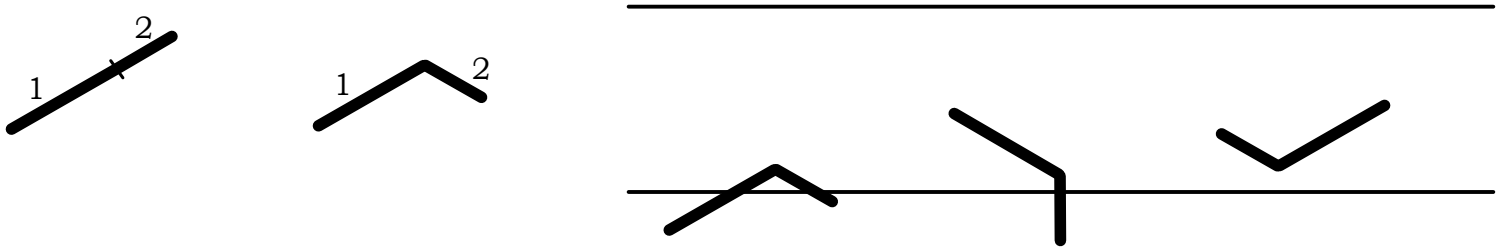

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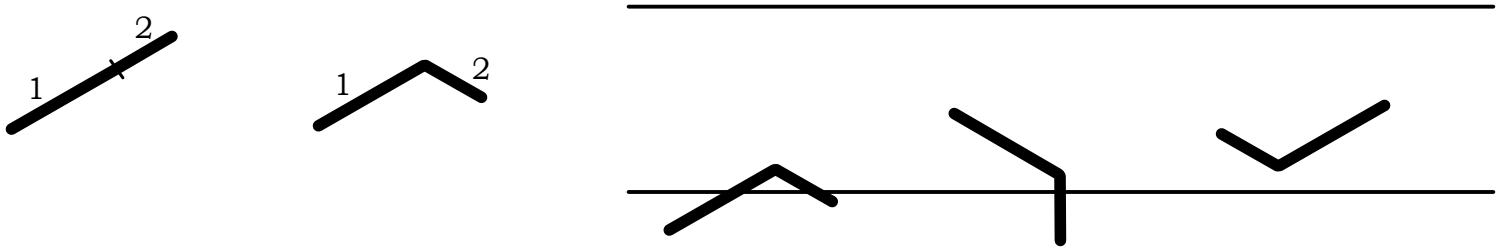

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- The answer to Buffon's problem is still equal to the expected number of intersections, even for the bent needle!
- Sanity test: fold the needle in half. It is now twice shorter, so the probability to have and intersection should be $p/2$. But each intersection is double, so $\mathbb{E}\hat{n} = 2 \times p/2 = p$. Good!

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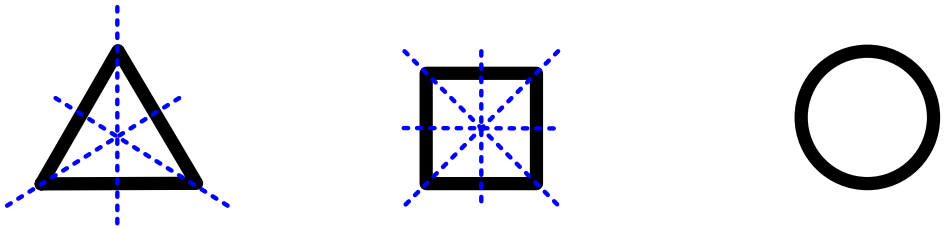
- What shape should we bend it into? The more symmetry the better!
- Triangle? Square? Hexagon? Circle!



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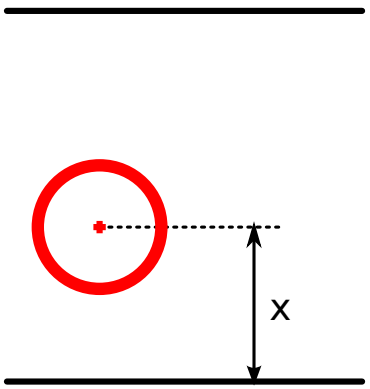

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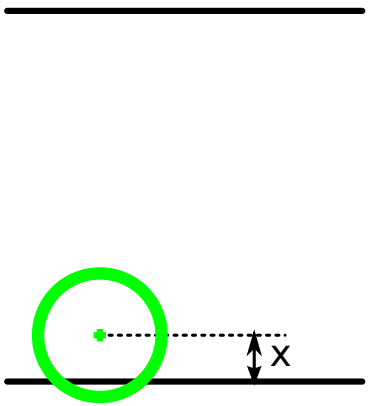

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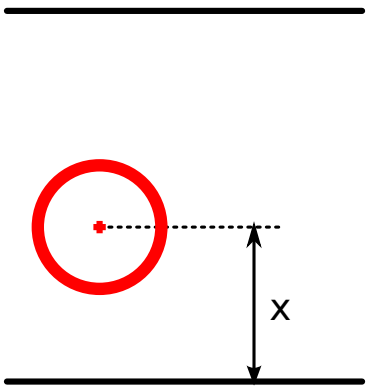

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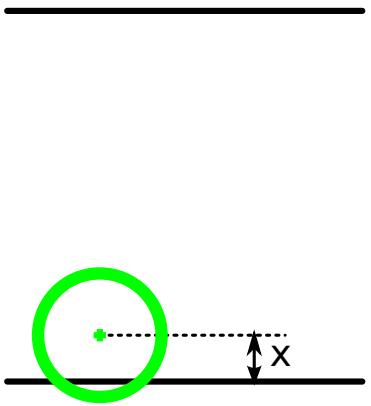

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- Let x be the distance from the center of the circle to the nearest line: $x \in [0, 1]$.
- We have an intersection when $x \leq R$. In fact, we have two!
- Calculate $\mathbb{E}n = 2 \times \frac{1}{\pi} + 0 \times \left(1 - \frac{1}{\pi}\right) = \frac{2}{\pi}$.

Easy as 1, 2, π !

Conclusions

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So the title was:

On π and Buffon's noodle.