Solving nonlinear recursions

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A general method to map a polynomial recursion on a matrix linear one is suggested. The solution of the recursion is represented as a product of a matrix multiplied by the vector of initial values. This matrix is product of *transfer* matrices whose elements depend only on the polynomial and not on the initial conditions. The method is valid for systems of polynomial recursions and for polynomial recursions of arbitrary order. The only restriction on these recurrent relations is that the highest-order term can be written in explicit form as a function of the lower-order terms (existence of a normal form). A continuous analog of this method is described as well.
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I. INTRODUCTION

Recurrent relations take a central place in various fields of science. For example, numerical solution of differential equations and models of evolution of a system involve, in general, recursions.

By now, only linear recursions could be solved[1–3] while even the simplest nonlinearity usually made an analytic solution impossible. A good example for this is a rather simple recursion, the logistic map, $y_{n+1}=\lambda y_n(1-y_n)$. The analysis of its behavior, while based on roundabout approaches, has revealed many unusual features.

In this paper we propose a new approach to the solution of polynomial recursions. It turns out that the coefficients of the i-th iteration of the polynomial depend linearly on the coefficients of the $(i-1)$-th iteration. Using this fact we succeed in writing down the general solution of the recursion.

To make this paper more readable we include some auxiliary material on linear recursions as well as an introductory example.

II. INTRODUCTORY EXAMPLE: LOGISTIC MAPPING

To demonstrate our approach we begin with the recursion equation known as the logistic mapping:

$$y_{n+1}=\lambda y_n(1-y_n) \quad \text{with} \quad y_0\equiv y. \tag{1}$$

Very recently it was shown by Rabinovich *et al.*[4] that the solution of this recursion is given by

$$y_n=\langle \mathbf{e}|\mathbf{T}^n|\mathbf{y}\rangle, \tag{2}$$

where $\mathbf{T}$ is a transfer matrix with elements

$$T_{jk}=(-1)^{k-j}\binom{j}{k-j}\lambda^j. \tag{3}$$

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The vectors $|\mathbf{y}\rangle$ and $\langle\mathbf{e}|$ are correspondingly a set of y's powers and the first ort defined as

$$|\mathbf{y}\rangle=\{y^j\}_{j=1}^{2^n} \quad \text{and} \quad \langle\mathbf{e}|=[\delta_{j1}]_{j=1}^{2^n}, \tag{4}$$

where δ_{jk} is the Kronecker symbol.

Equations (2) and (3) were derived in Ref. 4 by consideration of a branching process. However, knowing the representation of the solution (2) one can obtain the matrix elements (3) in a ''one-line'' way. Namely, we have to find a matrix $\mathbf{T}$ that transforms a column $\{y^j\}$ to a column $\{[\lambda y(1-y)]^j\}$. Expanding this last expression

$$[\lambda y(1-y)]^j=\sum_{i=0}^{j}(-1)^i\binom{j}{i}\lambda^j y^{j+i}=\sum_{k=j}^{2j}(-1)^{k-j}\binom{j}{k-j}\lambda^j y^k=\sum_{k=j}^{2j}T_{jk}y^k$$

and extending the last summation over all natural numbers {due to the vanishing of the binomials $\binom{j}{k-j}$ for k outside the interval $[j,2j]$} we immediately recover Eq. (3) for the elements of the matrix $\mathbf{T}$.

III. GENERAL CASE OF FIRST-ORDER POLYNOMIAL RECURSION

Here we consider a first-order recursion equation in its normal form

$$y_{n+1}=P(y_n), \tag{5}$$

where $P(x)$ is a polynomial of degree m:

$$P(x)=\sum_{k=0}^{m}a_kx^k, \quad a_m\neq 0. \tag{6}$$

Let $y_0\equiv y$ be an initial value for the recursion (5). We denote by $|\mathbf{y}\rangle$ the column vector of powers of y

$$|\mathbf{y}\rangle=\{y^j\}_{j=0}^{\infty}$$

and the vector $\langle\mathbf{e}|$ is a row vector

$$\langle\mathbf{e}|=[\delta_{j1}]_{j=0}^{\infty}.$$

It should be emphasized that j runs from 0, since in the general case $a_0\neq 0$. In this notation $\langle\mathbf{e}|\mathbf{y}\rangle$ is a scalar product that yields

$$\langle\mathbf{e}|\mathbf{y}\rangle=y. \tag{7}$$

Theorem: For any recursion of the type of Eq. (5) there exists a matrix $\mathbf{T}=\{T_{jk}\}_{j,k=0}^{\infty}$ such that

$$y_n=\langle\mathbf{e}|\mathbf{T}^n|\mathbf{y}\rangle. \tag{8}$$

Proof: For $n=0$ the statement of the theorem is valid [see Eq. (7)]. We introduce the column vector $|\mathbf{y}_1 \overset{\text{def}}{=} \{y_1^j\}_{j=0}^{\infty}$, where $y_1=P(y)$. Let $\mathbf{T}$ be a matrix such that

$$|\mathbf{y}_1\rangle=\mathbf{T}|\mathbf{y}\rangle. \tag{9}$$

The existence of this matrix will be proven later on. If such a matrix exists, then, analogically to Eq. (7), we have

$$y_1 = \langle \mathbf{e} | \mathbf{y}_1 \rangle = \langle \mathbf{e} | \mathbf{T} | \mathbf{y} \rangle.$$

Therefore, the statement of the theorem is true for $n=1$ as well.

Assume that Eq. (8) is valid for $n=l$ and any initial value y. Then y_{l+1} can be represented as $y_{l+1} = \langle \mathbf{e} | \mathbf{T}^l | \mathbf{y}_1 \rangle$, where $y_1 = P(y)$ is considered as a new initial value of the recursion. Then, using Eq. (9) one gets

$$y_{l+1} = \langle \mathbf{e} | \mathbf{T}^l | \mathbf{y}_1 \rangle = \langle \mathbf{e} | \mathbf{T}^l \mathbf{T} | \mathbf{y} \rangle = \langle \mathbf{e} | \mathbf{T}^{l+1} | \mathbf{y} \rangle.$$

To prove the existence of the matrix $\mathbf{T}$ we use $|\mathbf{y}_1\rangle \overset{\text{def}}{=} \{P^j(y)\}_{j=0}^{\infty}$. In turn, $P^j(y)$ is the jm-th degree polynomial

$$P^j(y) = \left(\sum_{i=0}^{m} a_i y^i \right)^j = \sum_{k=0}^{jm} T_{jk} y^k, \tag{10}$$

and we infer that $\mathbf{T} = \{T_{jk}\}_{j,k=0}^{\infty}$ obeys Eq. (9).

Note that for j and k satisfying $k \geqslant jm$ we have $T_{jk} \equiv 0$. Therefore, each row is finite (i.e., there is only a finite number of nonzero matrix elements in each row). This proves the existence of powers of $\mathbf{T}$ and completes the proof.

The method of this section can be generalized to an arbitrary analytic function in the right-hand side of Eq. (5).[5]

IV. SPECIAL CASES

A. The binomial case, $P(x) = a_p x^p + a_q x^q$

As one can see, in the general case elements of the matrix $\mathbf{T}$ have a form of rather complicated sums. However, they are degenerated to a fairly simple expression, when the polynomial (6) has only two terms. In this case one gets

$$P^j(y) = (a_p y^p + a_q y^q)^j = \sum_{i=0}^{j} \binom{j}{i} a_p^{j-i} a_q^i y^{p(j-i)+qi}.$$

Denoting

$$k = p(j-i) + qi, \quad i = l(k) = (q-p)^{-1}(k - pj),$$

we have

$$P^j(y) = \sum_{k=jp}^{jq} y^k \binom{j}{l(k)} a_p^{j-l(k)} a_q^{l(k)}.$$

Thus, the matrix elements T_{jk} are

$$T_{jk} = \binom{j}{l(k)} a_p^{j-l(k)} a_q^{l(k)}.$$

By substituting here $p=1$, $q=2$, $a_p = -a_q = \lambda$, we immediately recover the solution for the logistic map, Eq. (3).

B. The trinomial case, $P(x)=a_0+a_px^p+a_qx^q$, $a_0\neq 0$

Here, the transfer matrix $\mathbf{T}$ admits the following decomposition:

$$\mathbf{T}=\mathbf{A}\mathbf{T}_0,$$

where $\mathbf{T}_0$ is the matrix corresponding to the polynomial $P_0(x)=a_px^p+a_qx^q$ and $\mathbf{A}$ is an upper-triangular matrix. Indeed, let us consider $P_0(x)=a_px^p+a_qx^q$ and the corresponding matrix $\mathbf{T}_0$. It yields

$$\mathbf{T}_0|\mathbf{y}\rangle\overset{\text{def}}{=}|\mathbf{y}_1'\rangle=\{P_0^j(y)\}_{j=0}^{\infty}.$$

For the matrix $\mathbf{T}$ one gets

$$\mathbf{T}|\mathbf{y}\rangle\overset{\text{def}}{=}|\mathbf{y}_1\rangle=\{P(y)^j\}_{j=0}^{\infty},$$

$$P^j(y)=\sum_{i=0}^{j}\binom{j}{i}a_0^{j-i}(a_py^p+a_qy^q)^i.$$

Denoting in the last line $A_{ji}\equiv\binom{j}{i}a_0^{j-i}$ one obtains $|\mathbf{y}_1\rangle=\mathbf{A}|\mathbf{y}_1'\rangle=\mathbf{A}\mathbf{T}_0|\mathbf{y}\rangle=\mathbf{T}|\mathbf{y}\rangle$, and $\mathbf{T}=\mathbf{A}\mathbf{T}_0$.

V. NONCONSTANT COEFFICIENTS

As shown in Ref. 4 a generalization of Eq. (1),

$$y_{n+1}=\lambda_ny_n(1-y_n)\quad\text{with}\quad y_0\equiv y, \tag{11}$$

can be solved using a similar approach. The solution is

$$y_n=\langle\mathbf{e}|\mathbf{T}_n\cdots\mathbf{T}_2\mathbf{T}_1|\mathbf{y}\rangle, \tag{12}$$

where the matrix elements of $\mathbf{T}_i$ are now i-dependent:

$$(T_i)_{jk}=(-1)^{k-j}\binom{j}{k-j}(\lambda_i)^j. \tag{13}$$

The same argument is valid for an arbitrary recursion $y_{n+1}=P_n(n,y_n)$ and therefore solution Eq. (8) takes the form of Eq. (12) with the obvious changes ($a_0,\ldots,a_m$ become i-dependent functions) in the corresponding matrix elements.

VI. THE RICCATI RECURSION

This name is commonly used for the equation

$$y_{n+1}y_n+a_n'y_{n+1}+b_n'y_n+c_n'=0.$$

However, by a proper change of variable[1,2] this equation can be reduced to a linear one and then treated by conventional techniques. Here we shall be dealing with the following recursion:

$$y_{n+1}=a_n+b_ny_n+c_ny_n^2\quad\text{with}\quad y_0\equiv y.$$

This is a possible (asymmetric) discrete analog of the Riccati differential equation.[6] It is well known that the latter cannot be solved in quadratures.

The general results of the two previous sections can be employed to write down the solution of this recursion. Namely, the solution reads

$$y_n = \langle \mathbf{e} | \mathbf{T}_n \cdots \mathbf{T}_2 \mathbf{T}_1 | \mathbf{y} \rangle,$$

where the matrix $\mathbf{T}_i$ is a product of two matrices

$$\mathbf{T}_i = \mathbf{A}_i \mathbf{S}_i$$

with matrix elements

$$(\mathbf{A}_i)_{jk} = \binom{j}{k} a_i^{j-k} \quad \text{and} \quad (\mathbf{S}_i)_{jk} = \binom{j}{k-j} b_i^{2j-k} c_i^{k-j}.$$

VII. SYSTEM OF LINEAR FIRST-ORDER RECURSIONS

The next three sections deal with linear recursions. They are well known,[1,2] but we include those sections to help the understanding of subsequent sections, devoted to systems of nonlinear recursions.

The solution of a system of linear first-order recursions in the most general case is rather trivial, but for the sake of clarity we shall demonstrate it on a 2×2 homogeneous system

$$\begin{aligned} u_{n+1} &= (\lambda_{11})_n u_n + (\lambda_{12})_n v_n \quad \text{with } u_0 \equiv u, \\ v_{n+1} &= (\lambda_{21})_n u_n + (\lambda_{22})_n v_n \quad \text{with } v_0 \equiv v. \end{aligned} \tag{14}$$

Introducing the vector $\langle \mathbf{x}_n | = (u_n, v_n)$ and the matrix

$$\mathbf{\Lambda}_n = \begin{pmatrix} (\lambda_{11})_n & (\lambda_{12})_n \\ (\lambda_{21})_n & (\lambda_{22})_n \end{pmatrix},$$

one rewrites Eq. (14) as follows:

$$|\mathbf{x}_{n+1}\rangle = \mathbf{\Lambda}_n |\mathbf{x}_n\rangle$$

and, thus,

$$|\mathbf{x}_{n+1}\rangle = \mathbf{\Lambda}_n \ldots \mathbf{\Lambda}_0 |\mathbf{x}_0\rangle,$$

where $\langle \mathbf{x}_0 | = (u, v)$ is an initial vector.

Further generalization to a homogeneous system of N linear equations of first order is straightforward.

VIII. LINEAR EQUATION WITH NONCONSTANT COEFFICIENTS

The result of the previous section allows one to solve linear recursions of an arbitrary order with nonconstant coefficients. As usual, we start with the simplest case—a second-order equation

$$x_{n+1} + \lambda_n x_n + \mu_{n-1} x_{n-1} = 0. \tag{15}$$

Denoting $y_n \equiv \mu_{n-1} x_{n-1}$ we obtain the system

$$x_{n+1} = -\lambda_n x_n - y_n, \quad y_{n+1} = \mu_n x_n. \tag{16}$$

The solution of this equation is written as in the previous section but now

$$\mathbf{\Lambda}_n=\begin{pmatrix} -\lambda_n & -1 \\ \mu_n & 0 \end{pmatrix}$$

and the initial vector is $(x_1,\mu_0 x_0)$, where x_0 and x_1 are initial values of the recursion (15).

The method we used to transform Eq. (15) to Eq. (16) is well known in the theory of differential equations,[7] but it is useful for the simplest case of constant coefficients only.

Again, the generalization of (16) for a linear equation of arbitrary order is quite simple.

IX. SYSTEM OF LINEAR HIGHER-ORDER RECURSIONS

The generalization to higher orders is rather obvious. The simplest example is

$$\begin{aligned} &x_{n+1}+(\lambda_{11})_n x_n+(\lambda_{12})_n x_{n-1}+(\mu_{11})_n y_n+(\mu_{12})_n y_{n-1}=0,\\ &y_{n+1}+(\mu_{21})_n y_n+(\mu_{22})_n y_{n-1}+(\lambda_{21})_n x_n+(\lambda_{22})_n x_{n-1}=0. \end{aligned} \tag{17}$$

One proceeds as in the previous section with new variables $u_n=x_{n-1}$ and $v_n=y_{n-1}$. Then, the system (17) takes the form

$$x_{n+1}+(\lambda_{11})_n x_n+(\lambda_{12})_n u_n+(\mu_{11})_n y_n+(\mu_{12})_n v_n=0,$$

$$y_{n+1}+(\mu_{21})_n y_n+(\mu_{22})_n v_n+(\lambda_{21})_n x_n+(\lambda_{22})_n u_n=0,$$

$$u_{n+1}-x_n=0, \quad v_{n+1}-y_n=0,$$

i.e., the vector (x_n,y_n,u_n,v_n) is transformed by the transfer matrix

$$\mathbf{\Lambda}_n=\begin{pmatrix} -(\lambda_{11})_n & -(\mu_{11})_n & -(\lambda_{12})_n & -(\mu_{12})_n \\ -(\lambda_{21})_n & -(\mu_{21})_n & -(\lambda_{22})_n & -(\mu_{22})_n \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and subject to appropriate initial conditions.

X. SYSTEM OF NONLINEAR FIRST-ORDER RECURSIONS

Actually, very little is known about systems of nonlinear recursions.[8] We now extend our method of Sec. III to deal with systems of nonlinear equations. Let us demonstrate it on the following example:

$$\begin{aligned} &u_{n+1}=\lambda u_n(1-v_n) \quad \text{with } u_0\equiv u,\\ &v_{n+1}=\mu v_n(1-u_n) \quad \text{with } v_0\equiv v. \end{aligned} \tag{18}$$

Proceeding here as in Sec. III, we are checking the transformation of a product $u^j v^k$:

$$\begin{aligned} [\lambda u(1-v)]^j[\mu v(1-u)]^k &= \sum_{r,s} \lambda^j u^j (-1)^r \binom{j}{r} v^r \mu^k v^k (-1)^s \binom{k}{s} u^s \\ &= \sum_{p,q} u^p v^q (-1)^{(p-j)+(q-k)} \binom{j}{q-k}\binom{k}{p-j} \lambda^j \mu^k. \end{aligned} \tag{19}$$

We prefer to proceed with the aid of multidimensional matrices[9] as being the most natural way. However, a possibility of using traditional two-dimensional matrices also exists.[5]

Indeed, introducing here a four-dimensional matrix $\mathbf{T}$ with the elements

$$T_{jkpq}=(-1)^{(p-j)+(q-k)}\binom{j}{q-k}\binom{k}{p-j}\lambda^j\mu^k$$

(it can also be viewed as an ordinary matrix on the space of index pairs) we basically return to the familiar transfer-matrix construction but for more complex objects. Namely, we shall operate with a two-dimensional matrix $\mathbf{X}$, defined as a direct product of vectors $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$:

$$X_{jk}=u^j v^k.$$

Here the matrix $\mathbf{X}$ plays the same role as the vector $|\mathbf{y}\rangle$ in Sec. III. The four-dimensional matrix $\mathbf{T}$ is analogous to its two-dimensional relative $\mathbf{T}$. The multiplication of such matrices is defined rather naturally:

$$\mathbf{X}_1\mathbf{X}_2=\sum_{p,q}(X_1)_{pq}(X_2)_{pq}\,,\quad (\mathbf{TX})_{jk}=\sum_{p,q}T_{jkpq}X_{pq}\,,\quad (\mathbf{T}_1\mathbf{T}_2)_{jkpq}=\sum_{rs}(T_1)_{jkrs}(T_2)_{rspq}\,.$$

Note that the matrix analog of the scalar product of vectors is just a contraction, $\mathbf{X}_1\mathbf{X}_2$, in the tensor algebra nomenclature.

As in Sec. X, one can obtain the solution of the system in the form

$$u_n=\mathbf{E}_1\mathbf{T}^n\mathbf{X},\quad v_n=\mathbf{E}_2\mathbf{T}^n\mathbf{X},$$

where, as usual,

$$(\mathbf{E}_1)_{jk}=\delta_{1j}\delta_{0k}\,,\quad (\mathbf{E}_2)_{jk}=\delta_{0j}\delta_{1k}\,.$$

Further generalization of this approach is also rather simple. In the general case of m first-order nonlinear equations

$$x_{n+1}^{(i)}=P_i(x_n^{(1)},\ldots,x_n^{(m)}),\quad i=1,\ldots,m, \tag{20}$$

one has to consider the $2m$-dimensional transfer matrix $\mathbf{T}$. To construct it we are checking as before the product

$$P_{j_1,\ldots,j_m}\equiv P_1^{j_1}\cdots P_m^{j_m},$$

and the m-dimensional matrix $\mathbf{X}$, defined as a direct product of m vectors of initial values $|\mathbf{x}^{(1)}\rangle,\ldots,|\mathbf{x}^{(m)}\rangle$.

The polynomial $P_{j_1,\ldots,j_m}$ depends on m variables $x^{(1)},\ldots,x^{(m)}$ and therefore can be represented as

$$P_{(j_1,\ldots,j_m)}(x^{(1)},\ldots,x^{(m)})=\mathbf{T}_{(j_1,\ldots,j_m)}\mathbf{X},$$

where $\mathbf{T}_{j_1,\ldots,j_m}$ is a constant multidimensional matrix of coefficients of the polynomial $P_{j_1\ldots,j_m}$. This matrix $\mathbf{T}_{j_1,\ldots,j_m}$ is the $(j_1,\ldots,j_m)$-th m-section of the transfer matrix $\mathbf{T}$.

Then defining the matrix $\mathbf{E}_i$ by

$$(\mathbf{E}_i)_{j_1,\ldots,j_i,\ldots,j_m}=\delta_{0j_1}\ldots\delta_{1j_i}\ldots\delta_{0j_m}$$

one can write down the solution of the system in the form

$$x_n^{(i)} = \mathbf{E}_i \mathbf{T}^n \mathbf{X}.$$

XI. SYSTEM OF NONLINEAR HIGHER-ORDER RECURSIONS

We are not going to write down even the simplest example, but the scheme is quite obvious: introduction of new variables to bring each equation to the first-order structure and, then, construction of a transfer matrix (as in the two previous sections).

XII. CONTINUOUS ANALOG OF THE TRANSFER MATRIX

In this section we present a continuous generalization of our transfer matrix technique. We consider the general case, the multivariable function, $f:\mathbf{R}^n \to \mathbf{R}^n$. We do not try to establish the exact conditions for existence of all the functions involved, but merely describe the algorithm.

Let

$$F_{x\to s}[\varphi(\mathbf{x},\mathbf{t})] = (2\pi)^{-n} \int_{\mathbf{R}^n} \exp(-i\langle \mathbf{x},\mathbf{s}\rangle)\varphi(\mathbf{x},\mathbf{t})\, d\mathbf{x},$$

where $\langle \mathbf{x},\mathbf{s}\rangle$ is the scalar product of two real vectors $\mathbf{x}$ and $\mathbf{s}$, be the Fourier transform of the function $\varphi:\mathbf{R}^{2n}\to\mathbf{C}$, $\mathbf{s},\mathbf{x},\mathbf{t}\in\mathbf{R}^n$ and $F^{-1}_{s\to x}$ be the corresponding inverse Fourier transform.[10] Then, we define the transfer kernel of the function $f:\mathbf{R}^n\to\mathbf{R}^n$ as Fourier transform $T(\mathbf{t},\mathbf{s}) = F_{x\to s} \exp[i\langle f(\mathbf{x}),\mathbf{t}\rangle]$.

For example, let us consider the logistic map, $f(x) = \lambda x(1-x)$. Then the transfer kernel is the function[11]

$$T(t,s) = \int_{-\infty}^{\infty} \exp[-ixs + it\lambda x(1-x)]\, dx = \sqrt{\frac{\pi}{t\lambda}} \exp\left[-\frac{i\pi}{4} + \frac{i(t\lambda - s)^2}{4t\lambda}\right].$$

We define the product of the transfer kernels $S(\mathbf{t},\mathbf{s})$ and $T(\mathbf{t},\mathbf{s})$ of the functions $g(\mathbf{x})$ and $f(\mathbf{x})$ by

$$S\odot T(\mathbf{t},\mathbf{s}) \stackrel{\text{def}}{=} \int_{\mathbf{R}^n} S(\mathbf{t},\tau) T(\tau,\mathbf{s})\, d\tau.$$

Theorem: The product, $S\odot T(\mathbf{t},\mathbf{s})$, of the transfer kernels $S(\mathbf{t},\mathbf{s})$ and $T(\mathbf{t},\mathbf{s})$ is the transfer kernel of the composition $g\circ f(\mathbf{x})$, where $g\circ f(\mathbf{x}) \equiv g(f(\mathbf{x}))$.

Indeed, performing the inverse Fourier transform for the function $S\odot T(\mathbf{t},\mathbf{s})$ one gets

$$F^{-1}_{s\to x}[S\odot T(\mathbf{t},\mathbf{s})] = \int_{\mathbf{R}^n} S(\mathbf{t},\tau) F^{-1}_{s\to x}[T(\tau,\mathbf{s})]\, d\tau = \int_{\mathbf{R}^n} S(\mathbf{t},\tau) \exp[i\langle f(\mathbf{x}),\tau\rangle]\, d\tau$$
$$= F^{-1}_{\tau\to f(\mathbf{x})}[S(\mathbf{t},\tau)] = \exp[i\langle g\circ f(\mathbf{x}),t\rangle].$$

One can see that the product of the transfer kernels is defined in analogy to the matrix product and we can obtain the solution of the recursion (see Sec. III) in the form

$$y_n^{(j)} = -\ln\{F^{-1}_{s\to y}[\mathbf{T}^n(\mathbf{t},\mathbf{s})]|_{\mathbf{t}=i\mathbf{e}_j}\}$$

where $\mathbf{e}_j = \{\delta_{jk}\}_{k=1}^n$.

XIII. SUMMARY

In this paper we have presented a new method to obtain the solution of arbitrary polynomial recursions. The method has been generalized to systems of multivariable recursions and recursions of arbitrary order, in analogy to the solution of linear recursions, also presented in this paper.

Generally, the solution is obtained in the form of a matrix power, applied to the vectors of initial values. We have presented a way to construct such a matrix.

Famous and important examples, such as the logistic map and the Riccati recursion, have been considered and the corresponding matrices have been written down explicitly.

We also generalized the method in another direction. It is shown that instead of transfer matrix one can use transfer kernel which can be considered as a continuous matrix.[12]

While the investigation of the solutions found is beyond the scope of the paper this challenging task deserves a few words. For example, the logistic map solution (2) can be used to construct a generating function.[4] Unfortunately, this latter may have an essential singularity. Therefore, it is more natural to construct an exponential generating function $\phi(z) \equiv \Sigma_n^\infty (1/n!) y_n z^n = \langle \mathbf{e} | \exp(z\mathbf{T}) | \mathbf{y} \rangle$. Then, one can try to understand the parametric dependence of the logistic map asymptotics from a steepest descent of the Cauchy integral

$$y_n = \frac{1}{2\pi i} \oint_{C_0} \frac{\phi(z)}{z^{n+1}} \, dz,$$

where the contour C_0 includes the origin of coordinates.

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[1] R. P. Agarwal, *Difference Equations and Inequalities* (Dekker, New York, 1992).
[2] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations: Numerical Methods and Applications* (Academic, New York, 1988).
[3] F. B. Hildebrand, *Finite-Difference Equations and Simulations* (Prentice–Hall, Englewood Cliffs, NJ, 1968).
[4] S. Rabinovich, G. Berkolaiko, S. Buldyrev, A. Shehter, and S. Havlin, Physica A **218**, 457 (1995); Int. J. Bifurcation Chaos, in press, 1996.
[5] G. Berkolaiko, S. Rabinovich, and S. Havlin, submitted to Israel J. Math. (1996).
[6] D. Zwillinger, *Handbook of Differential Equations* (Academic, Boston, 1992), 2nd ed.
[7] R. E. Bellman, *Introduction to Matrix Analysis* (McGraw–Hill, New York, 1960).
[8] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications* (Kluwer, Dordrecht, 1993).
[9] N. P. Sokolov, *Introduction to the Theory of Multidimensional Matrices* (Naukova Dumka, Kiev, 1972) (in Russian).
[10] H. Bremermann, *Distributions, Complex Variables, and Fourier Transforms* (Addison–Wesley, Reading, MA, 1965).
[11] V. S. Vladimirov, *Equations of Mathematical Physics* (Dekker, New York, 1971).
[12] R. G. Cooke, *Infinite Matrices and Sequence Spaces* (Dover, New York, 1955).