NON-EXPONENTIAL STABILITY AND DECAY RATES IN NONLINEAR STOCHASTIC DIFFERENCE EQUATION WITH UNBOUNDED NOISES

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Abstract. We consider stochastic difference equation

$$x_{n+1} = x_n\Big(1 - hf(x_n) + \sqrt{h}g(x_n)\xi_{n+1}\Big), \quad n = 0, 1, \dots, \quad x_0 \in \mathbb{R}^1,$$

where functions f and g are nonlinear and bounded, random variables ξ_i are independent and $h > 0$ is a nonrandom parameter.

We establish results on asymptotic stability and instability of the trivial solution $x_n \equiv 0$. We also show, that for some natural choices of the nonlinearities f and g, the rate of decay of x_n is approximately polynomial: we find $\alpha > 0$ such that x_n decays faster than $n^{-\alpha+\varepsilon}$ but slower than $n^{-\alpha-\varepsilon}$ for any $\varepsilon > 0$.

It also turns out that if $g(x)$ decays faster than $f(x)$ as $x \to 0$, the polynomial rate of decay can be established exactly, $x_n n^\alpha \to \text{const}$. On the other hand, if the coefficient by the noise does not decay fast enough, the approximate decay rate is the best possible result.

1. Introduction

In this paper we address the questions of stability and the rate of decay of solutions of the difference equation

$$x_{n+1} = x_n\Big(1 - hf(x_n) + \sqrt{h}g(x_n)\xi_{n+1}\Big), \quad n = 0, 1, \dots, \quad x_0 \in \mathbb{R}^1, \tag{1}$$

where ξ_{n+1} are independent random variables. The functions f and g are nonlinear and are assumed to be bounded. The small parameter $h > 0$ usually arises as the step size in numerical schemes. Equation (1) may be viewed as stochastically perturbed version of a deterministic autonomous difference equation, where the random perturbation is state-dependent. In general, it does not have linear leading order spatial dependence close to the equilibrium. As a consequence of the non-hyperbolicity of the equilibrium, the convergence of solutions of (1) to its equilibrium zero cannot be expected to take place at an exponentially fast rate.

Similarly to deterministic difference equations, analyzing asymptotic behavior of stochastic difference equations is often harder (see [1, 2, 5, 12, 13, 16, 17, 18]) than analyzing their *differential* counterparts. Nonetheless, we feel it is very important to develop techniques and better understanding of the similarities and the differences between the two types of equations. In this, we are motivated by two principal reasons. Firstly, in many applied contexts the studied phenomena are intrinsically discrete (see, for example, [7]). Using continuous approximation can sometimes mask significant phenomena. Going in

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the other direction, numerical simulation of stochastic differential equations involves solving an associated difference equation. It is important to know whether the discretization can produce spurious behaviors and how can this be avoided. For example, one needs to study if the asymptotic behavior of the discretized equation is a faithful reproduction of the asymptotic behavior of the original equation. The corresponding property is called "A-stability" and it has been addressed in the stochastic context in [8], [11], [19].

In this paper we analyze sufficient and necessary conditions for solutions x_n to converge to zero as $n \to \infty$ ("stability") and the rate at which such convergence happens for different types of the nonlinearities f and g. Our results should be compared to an earlier work [4] (see also [3]) in which similar differential equations had been analyzed.

One of the technical difficulties arising in the study of stability of stochastic difference equations is dealing with unbounded noise. So far, many results have been only available for the case of bounded noises (e.g. [5]). Yet, one of the most applicable scenarios, discretization of the white noise, involves normally distributed (and thus unbounded) random variables. In this paper we develop a tool which is designed to overcome this difficulty. In particular, it is instrumental in proving the instability result (Theorem 8 in Section 4) in this paper and can also be used to prove instability in several related models (e.g. in [9, 10]). Section 3 is devoted to setting up this tool, which can be thought of as a discrete variant of the Itō formula, and proving the corresponding theorem (Theorem 5).

Armed with Theorem 5 we formulate and prove criteria for almost sure asymptotic stability and instability of solutions to equation (1). In Section 5 we concentrate on the decay rate of the solutions (assuming they converge to 0). The principal result here is the comparison theorem which provides implicit information on asymptotic behavior of solution x_n via the limit

$$\lim_{n\to\infty} \frac{\ln |x_n|}{\sum_{i=1}^{n} S(x_i)},$$

where $S(x)$ stands for either $g^2(x)$ or $|f(x)|$. In the special (but typical) cases of polynomially decaying f and g, we extract explicit information (see Corollary 18) on the decay rate of x_n in the form of the limit

$$\lim_{n\to\infty} \frac{\ln |x_n|}{\ln n} = -\lambda < 0.$$

The above limit allows one to conclude that the decay rate of x_n is of polynomial type. More precisely, for any $\epsilon > 0$, the following bound is valid *eventually* as $n \to \infty$,

$$n^{-\lambda-\epsilon} \le |x_n| \le n^{-\lambda+\epsilon}. \tag{2}$$

At this point a natural question arises: under what circumstances bound (2) can be strengthened to the exact power-law decay $x_n n^\lambda \to \text{const}$? This question is answered in Section 6. Heuristically, the answer can be described as follows. The convergence to zero can be caused either by the deterministic term $f(x)$ or by the noise term $g(x)\xi$, depending on the comparative speed of decay of $f(x)$ and $g(x)$ as x tends to 0. When $f(x)$ is dominant, the convergence of x_n happens at an exact power-law rate. On the other hand, if the noise term is significant, we show that an exact rate result is *impossible*, namely

$$\limsup_{n\to\infty} |x_n| n^\lambda = \infty, \qquad \text{and} \qquad \liminf_{n\to\infty} |x_n| n^\lambda = 0,$$

for some λ.

2. Auxiliary Definitions and Facts

In this section we give a number of necessary definitions and a lemmas we use to prove our results. A detailed exposition of the definitions and facts of the theory of random processes can be found in, for example, [20].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a complete filtered probability space. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables with $\mathbf{E}\xi_n = 0$. We assume that the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is naturally generated: $\mathcal{F}_{n+1} = \sigma\{\xi_{i+1} : i = 0, 1, ..., n\}$.

Among all the sequences $\{X_n\}_{n\in\mathbb{N}}$ of the random variables we distinguish those for which X_n are $\mathcal{F}_n$-measurable $\forall n \in \mathbb{N}$.

A stochastic sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be an *$\mathcal{F}_n$-martingale*, if $\mathbf{E}|X_n| < \infty$ and $\mathbf{E}\big[X_n\big|\mathcal{F}_{n-1}\big] = X_{n-1}$ for all $n \in \mathbb{N}$ *a.s.*

A stochastic sequence $\{\xi_n\}_{n\in N}$ is said to be an *$\mathcal{F}_n$-martingale-difference*, if $\mathbf{E}|\xi_n| < \infty$ and $\mathbf{E}\big(\xi_n\big|\mathcal{F}_{n-1}\big) = 0$ *a.s.* for all $n \in \mathbb{N}$.

We use the standard abbreviation "*a.s.*" for the wordings "almost sure" or "almost surely" throughout the text.

If $\{X_n\}_{n\in\mathbb{N}}$ is a martingale, in the form $X_n = \sum_{i=1}^n \rho_i$, then the *quadratic variation* of X is the process $\langle X\rangle$ defined by

$$\langle X_n\rangle = \sum_{i=1}^n \mathbf{E}[\rho_i^2|\mathcal{F}_{i-1}].$$

Three lemmas below are variants of martingale convergence theorems (see e.g. [20]).

Lemma 1. *If $\{X_n\}_{n\in\mathbb{N}}$ is a martingale, $X_n = \sum_{i=1}^n \rho_i$, then*

$$\left\{\omega : \sum_{i=1}^\infty \mathbf{E}[\rho_i^2|\mathcal{F}_{i-1}] < \infty\right\} \subseteq \{X_n \to\}.$$

Here $\{X_n \to\}$ denotes the set of all $\omega \in \Omega$ for which $\lim_{n\to\infty} X_n$ exists and is finite.

Lemma 2. *If $\{X_n\}_{n\in\mathbb{N}}$ is a martingale, $X_n = \sum_{i=1}^n \rho_i$, and*

$$\sum_{i=1}^\infty \mathbf{E}[\rho_i^2|\mathcal{F}_{i-1}] = \infty, \quad a.s.$$

Then, a.s.,

$$\frac{X_n}{\sum_{i=1}^n \mathbf{E}\left[\rho_i^2\middle|\mathcal{F}_{i-1}\right]} \to 0, \qquad n \to \infty.$$

Lemma 3. *If X_n is a non-negative martingale, then $\lim_{n\to\infty} X_n$ exists with probability 1.*

The following lemma is proved in [2].

Lemma 4. *Let $\{Z_n\}_{n\in\mathbb{N}}$ be a non-negative $\mathcal{F}_n$-measurable process, $\mathbf{E}|Z_n| < \infty$ $\forall n \in \mathbb{N}$, and*

$$Z_{n+1} \le Z_n + u_n - v_n + \nu_{n+1}, \quad n = 0, 1, 2, \dots,$$

where $\{\nu_n\}_{n\in\mathbb{N}}$ is an $\mathcal{F}_n$-martingale-difference, $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ are nonnegative $\mathcal{F}_n$-measurable processes and $\mathbf{E}|u_n|, \mathbf{E}|v_n| < \infty$ $\forall n \in \mathbb{N}$.

Then

$$\left\{\omega : \sum_{n=1}^\infty u_n < \infty\right\} \subseteq \left\{\omega : \sum_{n=1}^\infty v_n < \infty\right\} \bigcap \{Z \to\}.$$

3. Discretized Itō formula

Below we will make use of the notation $O(\cdot)$:

$$\alpha(r)=O(\beta(r)) \quad \text{as} \quad r\to r_0 \qquad \Leftrightarrow \qquad \limsup_{r\to r_0}\left|\frac{\alpha(r)}{\beta(r)}\right|<\infty. \tag{3}$$

Here r_0 can be a real number or $\pm\infty$ and the argument r can be both continuous and discrete. We will also use $o(\cdot)$:

$$\alpha(r)=o(\beta(r)) \quad \text{as} \quad r\to r_0 \qquad \Leftrightarrow \qquad \lim_{r\to r_0}\frac{\alpha(r)}{\beta(r)}=0.$$

Assumption 1. *We will make the following assumptions about the noise ξ_n:*

(i) *ξ_n are independent random variables satisfying*

$$\mathbf{E}\xi_n=0, \qquad \mathbf{E}\xi_n^2=1, \qquad \mathbf{E}|\xi_n|^3 \text{ are uniformly bounded,}$$

(ii) *the probability density functions $p_n(\xi)$ exist and satisfy*

$$x^3p_n(x)\to 0 \quad \text{as } |x|\to\infty \quad \text{uniformly in } n.$$

The following theorem can be thought of as a discretized relative of the Itō formula.

Theorem 5. *Consider $\varphi:\mathbb{R}^1\to\mathbb{R}^1$ such that there exists $\delta>0$ and $\widetilde{\varphi}:\mathbb{R}^1\to\mathbb{R}^1$ satisfying*

(i) *$\widetilde{\varphi}\equiv\varphi$ on $U_\delta=[1-\delta,1+\delta]$,*
(ii) *$\widetilde{\varphi}\in C^3(\mathbb{R}^1)$ and $|\widetilde{\varphi}'''(x)|\le M$ for some M and all $x\in\mathbb{R}^1$,*
(iii) *$\int_{\mathbb{R}^1}|\varphi-\widetilde{\varphi}|dx<\infty$.*

Let f and g be $\mathcal{F}$-measurable bounded random variables; ξ be an $\mathcal{F}$-independent random variable satisfying Assumption 1. Then

$$\mathbf{E}\left[\varphi\left(1+fh+g\sqrt{h}\xi\right)\Big|\,\mathcal{F}\right]=\varphi(1)+\varphi'(1)fh+\frac{\varphi''(1)}{2}g^2h+hfo(1)+hg^2o(1), \tag{4}$$

where the error terms $o(1)$ satisfy

(i) *if $|f|,|g|<K$ then $o(1)\to0$ as $h\to0$, uniformly in f and g,*
(ii) *if $h<H$ then $o(1)\to0$ as $f\to0$ and $g\to0$ uniformly in h.*

Proof. For brevity we will assume that f and g are constants and correspondingly use the non-conditional expectation, the proof of the general case being completely analogous.

The proof consists of two main parts. In the first part we derive formula (4) for $\mathbf{E}[\widetilde{\varphi}]$. In the second part we prove that, for our purposes, $\widetilde{\varphi}$ is a good approximation for φ. More precisely, we prove the following estimate for the error term,

$$\mathbf{E}[\varphi-\widetilde{\varphi}]=hg^2o(1).$$

Part A. By Taylor expansion,

$$\widetilde{\varphi}(1+x)=\widetilde{\varphi}(1)+\widetilde{\varphi}'(1)x+\frac{\widetilde{\varphi}''(1)}{2}x^2+\frac{\widetilde{\varphi}'''(\theta)}{6}x^3,$$

with θ lying between 0 and x. We substitute $x=fh+g\sqrt{h}\xi$ and take expectation. Using properties of ξ,

$$\mathbf{E}x=fh, \qquad \mathbf{E}x^2=f^2h^2+g^2h,$$

and therefore

$$\mathbf{E}\widetilde{\varphi}(x)=\widetilde{\varphi}(1)+\widetilde{\varphi}'(1)fh+\frac{\widetilde{\varphi}''(1)}{2}g^2h+\widetilde{\varphi}''(1)f^2h^2/2+\mathbf{E}[\widetilde{\varphi}'''(\theta)x^3]/6.$$

Because $\widetilde{\varphi}'''(\theta)$ is uniformly bounded, we can estimate, by expanding x^3,

$$\left|\mathbf{E}[\widetilde{\varphi}'''(\theta)x^3]/6\right| \leq M\mathbf{E}|x^3|/6 \leq fh^2O(fh^{1/2}) + g^2hO(gh^{1/2}) + g^2hO(fh).$$

This proves formula (4) for $\widetilde{\varphi}\left(1 + fh + g\sqrt{h}\xi\right)$.

Part B. We introduce the shorthand $c_1 = 1 + hf$ and $c_2 = \sqrt{h}g$ and seek an estimate for the error term

$$\Delta = \mathbf{E}\left[\varphi\left(c_1 + c_2\xi\right) - \widetilde{\varphi}\left(c_1 + c_2\xi\right)\right].$$

We have

$$\begin{aligned}\Delta &= \int_{\mathbb{R}^1} \left(\varphi\left(c_1 + c_2\xi\right) - \widetilde{\varphi}\left(c_1 + c_2\xi\right)\right) p(\xi)d\xi \\ &= \int_{\mathbb{R}^1} \left(\varphi(r) - \widetilde{\varphi}(r)\right) p\left(\frac{r - c_1}{c_2}\right) \frac{dr}{|c_2|} = \int_{\mathbb{R}^1\setminus U_\delta} \left(\varphi(r) - \widetilde{\varphi}(r)\right) p\left(\frac{r - c_1}{c_2}\right) \frac{dr}{|c_2|},\end{aligned}$$

where we introduced a new variable of integration, $r = c_1 + c_2\xi$, and excluded U_δ from the integration range because $\varphi(r) - \widetilde{\varphi}(r) = 0$ on U_δ.

Now we can estimate

$$\begin{aligned}|\Delta| &\leq \sup_{r\notin U_\delta}\left\{p\left(\frac{r - c_1}{c_2}\right)\frac{1}{|c_2|}\right\}\int_{\mathbb{R}^1}\left|\varphi(r) - \widetilde{\varphi}(r)\right| dr \\ &\leq |c_2|^2 C \sup_{r\notin U_\delta}\left\{p\left(\frac{r - c_1}{c_2}\right)\frac{1}{|c_2|^3}\right\} = hg^2 C \sup_{r\notin U_\delta}\left\{\frac{p(y)y^3}{(r - 1 - hf)^3}\right\},\end{aligned}$$

where

$$y = \frac{r - 1 - hf}{\sqrt{h}g}.$$

If *either* h is bounded and $f, g \to 0$ *or* $|f|$ and $|g|$ are bounded and $h \to 0$, it is easy to see that $y \to \infty$ uniformly on $r \in \mathbb{R}^1 \setminus U_\delta$. Since under the same conditions $(r - 1 - hf)^3$ is bounded away from zero, the assumption $p(y)y^3 \to 0$ implies

$$\sup_{r\notin U_\delta}\left\{\frac{p(y)y^3}{(r - 1 - hf)^3}\right\} = o(1),$$

and, therefore, $|\Delta| \leq hg^2 o(1)$. □

4. Stability and instability

We consider equation

$$x_{n+1} = x_n\left(1 + hf(x_n) + \sqrt{h}g(x_n)\xi_{n+1}\right), \quad n = 0, 1, \ldots, \tag{5}$$

with nonrandom initial value $x_0 \in \mathbb{R}^1$, and independent random variables ξ_n satisfying $\mathbf{E}\xi_n = 0$, $\mathbf{E}\xi_n^2 = 1$ for all $n \in \mathbb{N}$. The functions $g, f : \mathbb{R}^1 \to \mathbb{R}^1$ are nonrandom, continuous and bounded:

$$|g(u)|, |f(u)| \leq 1 \quad \forall u \in \mathbb{R}^1. \tag{6}$$

Theorem 6. *Let functions f and g be bounded and ξ_n satisfy Assumption 1. Let also*

$$\sup_{u\in\mathbb{R}^1\setminus\emptyset}\left\{\frac{2f(u)}{g^2(u)}\right\} = \beta < 1. \tag{7}$$

If h is small enough then $\lim_{n\to\infty} x_n(\omega) = 0$ a.s. where x_n is a solution to equation (5).

Remark 7. If $g(u) = 0$ for some $u \neq 0$, we consider (7) fulfilled iff $f(u) < 0$. Thus we impose no restrictions on $g(u)$ when $f(u) < 0$ for all nonzero u.

Proof. We raise equation (5) to a power $\alpha > 0$, which will be determined later,

$$|x_{n+1}|^\alpha = |x_n|^\alpha \left|1 + hf(x_n) + \sqrt{h}g(x_n)\xi_{n+1}\right|^\alpha,$$

and denote $z_n = |x_n|^\alpha$. We define $\phi_\alpha(y) = |y|^\alpha$, denote

$$\Phi_n = \mathbf{E}\left[\phi_\alpha\big(1 + hf(x_n) + \sqrt{h}g(x_n)\xi_{n+1}\big)\Big| \mathcal{F}_n\right] - 1, \tag{8}$$

$$\rho_{n+1} = z_n\Big(\phi_\alpha\big(1 + hf(x_n) + \sqrt{h}g(x_n)\xi_{n+1}\big) - \Phi_n - 1\Big) \tag{9}$$

and rewrite

$$z_{n+1} = z_n + z_n\Phi_n + \rho_{n+1}. \tag{10}$$

Here ρ_{n+1} is an $\mathcal{F}_{n+1}$-martingale-difference. Applying Theorem 5 to Φ_n we have: $\phi_\alpha(1) = 1$, $\phi'_\alpha(1) = \alpha$, $\phi''_\alpha(1) = \alpha(\alpha - 1)$ and, therefore,

$$\Phi_n = \alpha hf(x_n)\big(1 + o(1)\big) + \frac{\alpha(\alpha-1)}{2}hg^2(x_n)\big(1 + o(1)\big) \tag{11}$$

$$\leq \frac{1}{2}\alpha hg^2(x_n)\left(\beta + \alpha - 1 + o(1)\right), \tag{12}$$

where we used condition (7) to obtain the estimate on the second line. Since $\beta < 1$, we can choose α and h sufficiently small, so that $\Phi_n \leq 0$ for all x_n. Now we can apply Lemma 4 with $u_n = 0$ and $v_n = -z_n\Phi_n$ and conclude that z_n converges to some (possibly random) value z_∞.

To prove that z_∞ is a.s. 0 we assume the contrary: there exists $y \neq 0$ such that, for any $\delta > 0$, the probability that the limit of z_n lies in the interval $(y - \delta, y + \delta)$ is nonzero.

At the point y we either have $g(y) \neq 0$ and then $\Phi_n < 0$ by (12) or $g(y) = 0$ but then $f(y) < 0$ and again $\Phi_n < 0$ (now using (11)). By continuity, Φ_n remains bounded away from zero in a δ-neighborhood of y. Therefore $\sum_{n=1}^\infty z_n\Phi_n$ is divergent which contradicts Lemma 4. □

It turns out that condition (7) is close to being necessary for stability. For the same equation we now ask the opposite question: under what conditions on f and g solutions of (5) do *not* tend to zero.

Theorem 8. *Let f and g be bounded and ξ_n satisfy Assumption 1. Let also*

$$f(u) > 0 \quad \text{and} \quad g(u) \neq 0 \quad \text{when} \quad u \neq 0$$

and

$$\liminf_{u\to 0}\left\{\frac{2|f(u)|}{g^2(u)}\right\} > 1. \tag{13}$$

If x_n be a solution to equation (5) with an initial value $x_0 \in \mathbb{R}^1$ and h is small enough then $\mathbb{P}\left\{\lim_{n\to\infty} x_n(\omega) = 0\right\} = 0$.

Proof. Consider

$$\Phi_i = \mathbf{E}\left[\left|1 + hf(x_i) + \sqrt{h}g(x_i)\xi_{i+1}\right|^{-\alpha}\Big|\mathcal{F}_i\right] \tag{14}$$

with $\alpha < 1$ and with $\mathcal{F}_0$ being the trivial σ-algebra. Since $\varphi(x) = |x|^{-\alpha}$ is integrable around $x = 0$ and has bounded third derivative outside a neighborhood of 0, we can apply Theorem 5 to obtain

$$\Phi_i = 1 - \alpha hf(x_n)\big(1 + o(1)\big) + \frac{\alpha(\alpha+1)}{2}hg^2(x_n)\big(1 + o(1)\big).$$

In particular, Φ_i are finite. Therefore, we can form

$$M_n = \prod_{i=0}^{n-1} \frac{\left|1 - hf(x_i) + \sqrt{h}g(x_i)\xi_{i+1}\right|^{-\alpha}}{\Phi_i}.$$

Since $\mathbf{E}|M_n| = \mathbf{E}M_n = 1 < \infty$, M_n is a positive martingale, convergent by Lemma 3.

We now have the following representation for the solution x_n

$$|x_n|^\alpha = |x_0|^\alpha \prod_{i=0}^{n-1} \left|1 - hf(x_i) + \sqrt{h}g(x_i)\xi_{i+1}\right|^\alpha = |x_0|^\alpha M_n^{-1} \prod_{i=0}^{n-1} \Phi_i^{-1}. \tag{15}$$

Suppose now that the theorem is untrue, i.e. that $\mathbb{P}(\Omega_1) > 0$, where $\Omega_1 = \{\omega : \lim x_n(\omega) = 0\}$. Using condition (13), for each $\omega \in \Omega_1$ we can find $\delta \in (0,1)$ and $N(\omega, \delta)$ such that $g^2(x_n) < 2f(x_n)(1-\delta)$ for all $n > N(\omega, \delta)$.

Thus, for $n > N(\omega, \delta)$,

$$\Phi_i < 1 + \alpha hf(x_n)\Big(-1 + o(1) + (\alpha+1)(1-\delta)\big(1 + o(1)\big)\Big) < 1,$$

when h and α are small enough.

Applying this inequality to representation (15) we obtain

$$|x_n|^\alpha \geq |x_0|^\alpha M_n^{-1} \prod_{i=0}^{N(\omega,\delta)} \Phi_i^{-1}.$$

The only n-dependent factor on the right-hand side is M_n^{-1} which tends to a nonzero limit by Lemma 3. All other factors being nonzero as well, we conclude that x_n^α remains bounded away from 0, which contradicts our definition of Ω_1. □

Remark 9. A more refined analysis of the factors in representation (15) allows to strengthen the conclusion of Theorem 8 to $\mathbb{P}\{\liminf_{n\to\infty} x_n(\omega) = 0\} = 0$. The proof of this statement, however, is unpleasantly technical and we leave it out.

5. Decay Rate

In this section we establish results on the a.s. decay rate of solutions x_n of (5).

The first subsection contains some variations of the classical Toeplitz lemma from analysis. In the second subsection we present a result about asymptotic behavior of $\ln x_n^2$ in two cases: when $\lim_{u\to 0} \frac{f(u)}{g(u)^2} = L$ and when $\lim_{u\to 0} \frac{f(u)}{g(u)^2} = -\infty$. These conditions include, but are weaker than, the sufficient conditions given by Theorem 6 for the stability of x_n. For this reason we explicitly assume that $x_n \to 0$ when we discuss the rate of decay.

5.1. Variations on Toeplitz Lemma.

In this section we state Toeplitz Lemma and prove one of its corollaries. The version of Toeplitz Lemma we need is taken from ([20], p. 390).

Lemma 10 (Toeplitz Lemma). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $\sum_{i=1}^\infty a_i$ diverges. If $\kappa_n \to \kappa_\infty$ as $n \to \infty$ then*

$$\lim_{n\to\infty} \frac{\sum_{i=0}^n a_i\kappa_i}{\sum_{i=0}^n a_i} = \kappa_\infty.$$

We will use the following 2 corollaries of Toeplitz Lemma.

Lemma 11. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers with $\sum_{i=1}^n a_i \to \infty$ when $n \to \infty$. Then*

$$\lim_{n\to\infty} \frac{b_n}{a_n} = c \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\sum_{i=1}^n b_i}{\sum_{i=1}^n a_i} = c.$$

Also,

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n b_i}{\sum_{i=1}^n a_i} = \infty \quad \Rightarrow \quad \limsup_{n\to\infty} \frac{b_n}{a_n} = \infty.$$

Proof. The statement follows from the representation

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n b_i}{\sum_{i=1}^n a_i} = \lim_{n\to\infty} \frac{\sum_{i=1}^n \frac{b_i}{a_i} a_i}{\sum_{i=1}^n a_i},$$

and Toeplitz Lemma. □

The following lemma is useful for extracting information about a sequence $\{y_n\}_{n\in\mathbb{N}}$ if what is known is given in an implicit form such as $f(y_n)(y_{n+1} - y_n) \to c$ for some function f.

Lemma 12. *Let $c > 0$, let $f : \mathbb{R}^1{}_+ \to \mathbb{R}^1{}_+$ be monotonous continuous function. Let $\{y_n\}_{n\in\mathbf{N}}$ be a positive increasing sequence such that $\lim_{n\to\infty} \frac{f(y_n)}{f(y_{n-1})} = 1$ and $\lim_{n\to\infty} \Delta y_n = 0$, where $\Delta y_n = y_{n+1} - y_n$.*

(i) *If $\lim_{n\to\infty} f(y_n)\Delta y_n = c$, then $\lim_{n\to\infty} \frac{1}{n}\int_{y_0}^{y_n} f(u)du = c$.*

(ii) *If $f(y_k)\Delta y_k \le c$, $k \le n$, then $\frac{1}{n}\int_{y_0}^{y_n} f(u)du \le c$.*

(iii) *If $f(y_k)\Delta y_k \ge c$, $k \le n$, then $\frac{1}{n}\int_{y_0}^{y_n} f(u)du \ge c$.*

Proof. Since in case (i) we also have $\lim_{n\to\infty} f(y_{n+1})\Delta y_n = c$, by Toeplitz Lemma we therefore conclude that

$$\frac{1}{n}\sum_{i=0}^{n-1} f(y_i)\Delta y_i \to c, \quad \frac{1}{n}\sum_{i=0}^{n} f(y_{i+1})\Delta y_i \to c \quad \text{as} \quad n \to \infty. \tag{16}$$

If f is decreasing, then by geometrical consideration it is clear that

$$\sum_{i=0}^{n-1} f(y_i)\Delta y_i \ge \int_{y_0}^{y_n} f(u)du \ge \sum_{i=1}^{n} f(y_{i+1})\Delta y_i, \tag{17}$$

and the result follows from (16). If f is an increasing function, then we reverse inequalities in (17).

To prove (ii) in case of decreasing f we note that

$$\int_{y_0}^{y_n} f(u)du \le \sum_{i=0}^{n-1} f(y_i)\Delta y_i \le cn.$$

Case of increasing f and (iii) are analogous. □

Corollary 13. *Let $c, \gamma > 0$ be non-random numbers. Let $\{y_n\}_{n\in\mathbb{N}}$ be a positive increasing sequence and let*

$$\lim_{n\to\infty} \Delta y_n = 0, \quad \lim_{n\to\infty} y_n^\gamma \Delta y_n = c. \tag{18}$$

Then

$$\frac{y_n}{n^{\frac{1}{1+\gamma}}} \to \big(c(1+\gamma)\big)^{\frac{1}{1+\gamma}}.$$

Proof. We put $f(u) = u^\gamma$ for $u \geq 0$, and note that $y_n \to \infty$ by (18). Therefore

$$\lim_{n\to\infty} \frac{f(y_n)}{f(y_{n-1})} = \lim_{n\to\infty} \left(\frac{y_n}{y_{n-1}}\right)^\lambda = 1.$$

Then by Lemma 12

$$c = \lim_{n\to\infty} \frac{1}{n} \int_{y_0}^{y_n} f(u)du = \lim_{n\to\infty} \frac{y_{n+1}^{\gamma+1} - y_0^{\gamma+1}}{n(\gamma+1)} = \lim_{n\to\infty} \frac{y_{n+1}^{\gamma+1}}{n(\gamma+1)},$$

and result follows. □

5.2. **A comparison theorem.**

Theorem 14. *Suppose that f and g are bounded with $f(0) = g(0) = 0$ and the random variables ξ_n satisfy Assumption 1. Assume, further, that $x_n \to 0$ a.s., where x_n is a solution of (5).*

a) *If*

$$\lim_{u\to 0} \frac{f(u)}{g^2(u)} = L, \quad L \in \mathbb{R}^1, \tag{19}$$

then

$$\lim_{n\to\infty} \frac{\ln|x_n|}{\sum_{i=1}^n g^2(x_i)} = hL - h/2. \tag{20}$$

b) *If $f(u) \leq 0$ in a neighborhood of $u = 0$ and*

$$\lim_{u\to 0} \frac{|f(u)|}{g(u)^2} = \infty, \tag{21}$$

then

$$\lim_{n\to\infty} \frac{\ln|x_n|}{\sum_{i=1}^n |f(x_i)|} = -h.$$

Remark 15. If the initial value x_0 is non-zero and the distributions of ξ_n are non-atomic, the solution x_n is a.s. non-zero for any n.

Remark 16. Theorem 8 imposes restrictions on possible values of L. To have convergent x_n we must have $L \leq 1/2$. Then, in equation (20), $-h + 2Lh$ is non-positive which one would expect with $x_n \to 0$.

Proof. We apply logarithm to both parts of equation (5) to obtain the following representation of the solution of the recursion

$$\ln|x_k| = \ln|x_0| + \sum_{i=1}^{k-1} \ln\left|1 + hf(x_i) + \sqrt{h}g(x_i)\xi_{i+1}\right|. \tag{22}$$

We set

$$\begin{aligned} \lambda_{i+1} &= \ln\left|1 + hf(x_i) + \sqrt{h}g(x_i)\xi_{i+1}\right|, \\ \Phi_i &= \mathbf{E}\left[\lambda_{i+1} | \mathcal{F}_i\right], \\ d_{i+1} &= \lambda_{i+1} - \Phi_i. \end{aligned}$$

The expectation Φ_i can be estimated by Theorem 5,

$$\Phi_i = hf(x_i)\big(1 + o(1)\big) - h\frac{g^2(x_i)}{2}\big(1 + o(1)\big). \tag{23}$$

On the other hand, it is easy to see that d_{i+1} is martingale-difference. Using Theorem 5 once again (now with $\varphi = \ln^2|1+x|$) we can estimate

$$\begin{aligned} \mathbf{E}\left[d_{i+1}^2\middle|\mathcal{F}_i\right] &= \mathbf{E}\left[\lambda_{i+1}^2\middle|\mathcal{F}_i\right] - \Phi_i^2 = hf(x_i)o(1) + hg^2(x_i)\big(1+o(1)\big) - \Phi_i^2 \\ &= hg^2(x_n)\big(1+o(1)\big), \end{aligned} \tag{24}$$

where to get to the final result we used condition (19) and estimate (23). We remind the reader that $o(1) \to 0$ as $i \to \infty$.

Part a). Now we would like to apply the Toeplitz Lemma (Lemma 10) to the left-hand side of (20), where $\ln|x_n|$ is expanded as in (22). To apply Toeplitz Lemma we need to show that the event

$$\Lambda = \{\omega \in \Omega : \sum_{i=1}^{\infty} g^2(x_i) < \infty\}$$

has zero probability. Since, by assumption, $x_i \to 0$ a.s., we have $f(x_i) = O(g^2(x_i))$ as $i \to \infty$. Thus we conclude that on (almost all of) Λ the series $\sum_{i=1}^{\infty} |f(x_i)|$ is convergent too. Consequently, the series $\sum_{i=1}^{\infty} \Phi_i$, $\sum_{i=1}^{\infty} \Phi_i^2$ and $\sum_{i=1}^{\infty} \mathbf{E}\left[d_{i+1}^2\middle|\mathcal{F}_i\right]$ are all absolutely convergent on Λ.

We now rewrite (22) as

$$\ln|x_k| = \ln|x_0| + \sum_{i=1}^{k-1} \Phi_i + \sum_{i=1}^{k-1} d_{i+1}$$

and notice that the right-hand side is convergent. This, however, contradicts our assumption that $x_i \to 0$ a.s. We conclude that Λ is a zero-probability event. As a consequence, we obtain that the characteristic of the martingale $\sum_{i=1}^{n} d_{i+1}$ is divergent too, see equation (24).

We can now write

$$\lim_{n\to\infty} \frac{\ln|x_n|}{\sum_{i=1}^{n} g^2(x_i)} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} \Phi_i}{\sum_{i=1}^{n} g^2(x_i)} + \lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} d_{i+1}}{\sum_{i=1}^{n} g^2(x_i)}. \tag{25}$$

The first limit on the right can be evaluated using (23) and Lemma 11,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} \Phi_i}{\sum_{i=1}^{n} g^2(x_i)} = h \lim_{i\to\infty} \frac{f(x_i)\big(1+o(1)\big)}{g^2(x_i)} - \frac{h}{2} \lim_{i\to\infty} \frac{g^2(x_i)\big(1+o(1)\big)}{g^2(x_i)} = hL - h/2.$$

The second limit in (25) is represented as

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} d_{i+1}}{\sum_{i=1}^{n} g^2(x_i)} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} d_{i+1}}{\sum_{i=1}^{n-1} \mathbf{E}\left[d_{i+1}^2\middle|\mathcal{F}_i\right]} \lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} \mathbf{E}\left[d_{i+1}^2\middle|\mathcal{F}_i\right]}{\sum_{i=1}^{n} g^2(x_i)}.$$

While the second limit on the right is finite (equal to h to be precise), the first one is zero by Lemma 2.

Part b). We follow the proof of part a) with $f(x_i)$ instead of $g^2(x_i)$. By a similar reasoning we conclude that the series $\sum_{i=1}^{\infty} |f(x_i)|$ is divergent. Indeed, if it were not so, the series $\sum_{i=1}^{\infty} g^2(x_i)$ would be convergent too, since $g^2(x_i)/|f(x_i)| \to 0$. We then conclude that the series $\sum_{i=1}^{\infty} \Phi_i$, $\sum_{i=1}^{\infty} \Phi_i^2$ and $\sum_{i=1}^{\infty} \mathbf{E}\left[d_{i+1}^2\middle|\mathcal{F}_i\right]$ are all absolutely convergent and therefore $\ln|x_k|$ converges to a finite limit. This contradicts our assumptions.

We write

$$\lim_{n\to\infty} \frac{\ln|x_n|}{\sum_{i=1}^{n} |f(x_i)|} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} \Phi_i}{\sum_{i=1}^{n} |f(x_i)|} + \lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} d_{i+1}}{\sum_{i=1}^{n} |f(x_i)|}, \tag{26}$$

and evaluate the first limit using Lemma 11,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} \Phi_i}{\sum_{i=1}^{n} g^2(x_i)} = h \lim_{i\to\infty} \frac{f(x_i)\big(1+o(1)\big)}{|f(x_i)|} - \frac{h}{2} \lim_{i\to\infty} \frac{g^2(x_i)\big(1+o(1)\big)}{|f(x_i)|} = -h - \frac{h}{2} \times 0.$$

To conclude the proof, the second limit in (26) is evaluated to zero as in part a). □

5.3. **Rate of decay of** $\ln|x_n|$**.** Theorem 14 provides some information on the decay of solutions x_n to zero, but does it in a rather implicit way. We will now show how one can extract an explicit estimate on the decay of $\ln|x_n|$ as a function of n.

Consider the following example. Let it be given that $x_n \to 0$ a.s. as $n \to \infty$ (see Theorem 6 for a set of sufficient conditions) and let condition (19) be satisfied. Assume that the function $g(u)$ behaves like a power of u around zero,

$$\lim_{u\to 0} \frac{g^2(u)}{u^{\mu_g}} = \text{const}.$$

Then, using the following lemma we can conclude that

$$\lim_{n\to\infty} \frac{\ln|x_n|}{\ln\left(n^{-1/\mu_g}\right)} = 1.$$

Lemma 17. *Let* $\lambda > 0$ *and* x_n *be a positive sequence satisfying*

$$\lim_{n\to\infty} x_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\ln x_n}{\sum_{i=1}^{n} x_i^\lambda} = -b < 0. \tag{27}$$

Then

$$\lim_{n\to\infty} \frac{\ln x_n}{\ln n} = -1/\lambda.$$

Proof. We will prove the lemma for $\lambda = 1$; the general case would follow with the change $x_i \mapsto x_i^\lambda$. Consider a new variable $y_n = \sum_{i=1}^{n} x_i$. With $\Delta y_n = y_n - y_{n-1}$ we can rewrite the second condition in (27) as

$$\lim_{n\to\infty} \frac{\ln \Delta y_n}{y_n} = -b.$$

By applying the definition of limit, given a small $\varepsilon > 0$ we can find N such that for all $n \geq N$

$$\frac{\ln \Delta y_n}{y_n} \leq -(b-\varepsilon) < 0.$$

This can be transformed into

$$1 \geq e^{(b-\varepsilon)y_n} \Delta y_n.$$

Applying Lemma 12, (ii), we obtain

$$\frac{1}{n(b-\varepsilon)} \left[e^{(b-\varepsilon)y_n} - e^{(b-\varepsilon)y_N}\right] = \frac{1}{n} \int_{y_N}^{y_n} e^{(b-\varepsilon)u} du \leq 1.$$

Solving for y_n gives

$$y_n \leq \frac{\ln\left[C_1 n + C_2\right]}{(b-\varepsilon)},$$

where C_1 and C_2 are constant with respect to n. Sending n to infinity we obtain

$$\limsup_{n\to\infty} \frac{y_n}{\ln n} \leq \frac{1}{(b-\varepsilon)}. \tag{28}$$

To obtain a similar bound from below we notice that

$$\frac{y_n}{y_{n-1}} = \frac{y_{n-1} + x_n}{y_{n-1}} \to 1,$$

and therefore

$$\lim_{n\to\infty} \frac{\ln \Delta y_n}{y_{n-1}} = -b.$$

Retracing our steps with $+\varepsilon$ instead of $-\varepsilon$ and applying Lemma 12, (iii), produce

$$\frac{1}{n(b+\varepsilon)}\left[e^{(b+\varepsilon)y_n} - e^{(b+\varepsilon)y_N}\right] = \frac{1}{n}\int_{y_N}^{y_n} e^{(b+\varepsilon)u}du \geq 1$$

and, ultimately,

$$\liminf_{n\to\infty} \frac{y_n}{\ln n} \geq \frac{1}{(b+\varepsilon)}. \tag{29}$$

Since ε was arbitrary, we obtain from (28) and (29)

$$\lim_{n\to\infty} \frac{y_n}{\ln n} = \frac{1}{b},$$

and, using the definition of y_n and condition (27),

$$\lim_{n\to\infty} \frac{\ln x_n}{\ln n} = \lim_{n\to\infty} \frac{\ln x_n}{y_n}\frac{y_n}{\ln n} = -b\frac{1}{b} = -1.$$

□

Next we formulate a corollary which extends and formalizes the discussion at the start of the present section.

Corollary 18. *Suppose that f and g are bounded with $f(0) = g(0) = 0$ and the random variables ξ_n satisfy Assumption 1. Assume, further, that $x_n \to 0$ a.s., where x_n is a solution of (5). If one of the following conditions is fulfilled,*

a)

$$\lim_{u\to 0} \frac{f(u)}{|u|^\lambda} = c < 0, \qquad \lim_{u\to 0} \frac{f(u)}{g(u)^2} = -\infty, \tag{30}$$

or

b)

$$\lim_{u\to 0} \frac{g^2(u)}{|u|^\lambda} = c > 0, \qquad \lim_{u\to 0} \frac{f(u)}{g(u)^2} = L < \frac{1}{2},$$

then

$$\lim_{n\to\infty} \frac{\ln|x_n|}{\ln n} = -\frac{1}{\lambda}.$$

Proof. For the solution x_n we have

$$\lim_{n\to\infty} \frac{\ln|x_n|}{\sum_{i=1}^n |x_i|^\lambda} = \lim_{n\to\infty} \frac{\ln|x_n|}{\sum_{i=1}^n f(x_i)} \lim_{n\to\infty} \frac{\sum_{i=1}^n f(x_i)}{\sum_{i=1}^n |x_i|^\lambda} = h \lim_{n\to\infty} \frac{f(x_i)}{|x_i|^\lambda} = hc < 0,$$

where the first limit was calculated using Theorem 14 and the second was done using Lemma 11 and condition (30). Now we apply Lemma 17 to finish the proof of Part a). The proof of Part b) is analogous. □

Remark 19. Relation

$$\lim_{n\to\infty} \frac{\ln|x_n|}{\ln n} = -\frac{1}{\lambda} < 0, \quad \text{a.s.}$$

implies that for every $\varepsilon > 0$ there exists $N = N(\varepsilon, \omega)$ such that a.s. for all $n \geq N(\varepsilon, \omega)$

$$n^{-\frac{1}{\lambda}-\varepsilon} \leq |x_n| \leq n^{-\frac{1}{\lambda}+\varepsilon}.$$

We would also like to mention that Lemma 17 can be extended to include other forms of the functions $f(u)$ and $g^2(u)$ around the origin. We give this result here without the proof, which is a simple extension of the proof of Lemma 17 (see also Theorem 5 of [4]).

Lemma 20. *Let x_n be a positive sequence satisfying*

$$\lim_{n\to\infty} x_n = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\ln|x_n|}{\sum_{i=1}^n |f(x_i)|} = -b < 0.$$

Suppose there exists a function $a(u)$ satisfying

- *$a(u)$ is monotone increasing,*
- *$|f(u)|/a(u) \to 1$ as $u \to 0+$,*
- *the function $A(z)$, defined by $A(z) = \int_z^1 \frac{du}{ua(u)}$, obeys*

  $$\lim_{y\to\infty} \frac{\ln\left[A^{-1}(y - y^*)\right]}{\ln\left[A^{-1}(y)\right]} = 1$$

  for any constant y^*.

Then

$$\lim_{n\to\infty} \frac{\ln|x_n|}{\ln\left[A^{-1}(n)\right]} = 1.$$

6. Exact Rate of Decay

In this section we derive the exact decay rate (or prove its absence) in the case when f and g have power-law behavior:

$$f(u) \sim -a_f|u|^{\mu_f}, \quad g^2(u) \sim a_g|u|^{\mu_g} \quad \text{as} \quad u \to 0, \tag{31}$$

where $\mu_f, \mu_g, a_g > 0$ and $a_f \neq 0$.

We will assume that

$$\lim_{u\to 0} \frac{2f(u)}{g^2(u)} < 1. \tag{32}$$

and that $x_n \to 0$.

The assumptions above ensure that the conditions of Corollary 18 are satisfied, which gives us a preliminary estimate on the rate of decay of x_n (see Remark 19). The aim of this section is to strengthen the result of Remark 19 to the result of the type $x_n n^\lambda \to \text{const}$ or to show that such strengthening is impossible.

Remark 21. It is clear that if f and g are given by (31) then condition (32) holds in the following three cases:

(i) $\mu_f > \mu_g$,
(ii) $\mu_f = \mu_g$ and $-2a_f < a_g$,
(iii) $\mu_f < \mu_g$ and $a_f > 0$.

We will also need to strengthen Assumption 1 about the noise ξ_n:

Assumption 2. *Let ξ_n be independent random variables. We assume that $\mathbf{E}\xi_n = 0$, $\mathbf{E}\xi_n^2 = 1$ and for each $m \in \mathbb{N}$ there is a $C(m) > 0$ such that*

$$\mathbf{E}[|\xi_n|^m] \le C(m). \tag{33}$$

An important example of the noise satisfying Assumption 2 are the i.i.d. normal ξ_n. The condition (33) implies that the large fluctuations of ξ_n grow slower than any power.

Lemma 22. *Suppose that* (33) *holds. Then for every fixed* $\varepsilon > 0$

$$\lim_{n\to\infty} n^{-\varepsilon}|\xi_n| = 0, \quad a.s.$$

This lemma is a direct consequence of (33) and the Borel-Cantelli lemma.

Corollary 23. *From power-law decay of* x_n *(Remark 19) and Lemma 22 we conclude that for every fixed* $\varepsilon > 0$

$$\lim_{n\to\infty} |x_n|^{\varepsilon}|\xi_n| = 0, \quad a.s.$$

In particular, $g(x_n)\xi_{n+1} \to 0$ *as* $n \to \infty$.

For the sake of simplicity everywhere below we are going to hide h and $\sqrt{h}$. In other words we let

$$f := hf, \quad g := \sqrt{h}g.$$

Remark 24. Since $x_n \to 0$, $f(x_n) \to 0$ and $g(x_n)\xi_{n+1} \to 0$ as $n \to \infty$, after some random number of steps the bracket

$$(1 + f(x_n) + g(x_n)\xi_{n+1})$$

becomes positive. In particular, it means that solution x_n eventually stops changing sign.

6.1. **Main Results.** It turns out that, in the situation described in case (iii) of Remark 21, the decay rate is exact.

Theorem 25. *Suppose Assumption 2 holds. Let* $a_f > 0$ *and* $\mu_f < \mu_g$*. Then*

$$\lim_{n\to\infty} |x_n| n^{\frac{1}{\mu_f}} = \left[\frac{1}{a_f\mu_f}\right]^{\frac{1}{\mu_f}}, \quad a.s.$$

The intuitive reason for the above behavior is that the convergence under the conditions of the Theorem is dominated by the deterministic terms of the recursion and the rate coincides with the corresponding deterministic ($\xi_n \equiv 0$) rate.

On the other hand, if conditions of cases (i) or (ii) of Remark 21 are met, then x_n undergoes large oscillations around the power law decay. This is because the convergence is induced by the noise term, which is now significant. More detailed explanations are given in Remark 28 below and, of course, in the proofs in Sections 6.3 and 6.4.

Theorem 26. *Suppose that Assumption 2 holds. Let either* $\mu_f > \mu_g$ *or* $\mu_f = \mu_g$ *and* $-2a_f < a_g$ *hold (cases (i)and (ii) of Remark 21 correspondingly). Suppose moreover that* $g(u) \geq 0$ *for* $u > 0$ *and there exists some* $r > 0$ *such that*

$$\frac{1}{\sqrt{a_g}} u^{-\mu_g/2} g(u) = 1 + o(u^r) \quad as \quad u \to +0. \tag{34}$$

Then

$$\limsup_{n\to\infty} |x_n| n^{\frac{1}{\mu_g}} = \infty, \quad a.s. \tag{35}$$

$$\liminf_{n\to\infty} |x_n| n^{\frac{1}{\mu_g}} = 0, \quad a.s. \tag{36}$$

6.2. **Main Construction and an Outline of the Proofs.** By squaring both part of (5) with $f := hf, \quad g := \sqrt{h}g$, we obtain the equation

$$x_{n+1}^2 = x_n^2\left(1 + F_n + R_{n+1}\right), \quad n = 0, 1, \dots, \tag{37}$$

where

$$F_n = 2f(x_n) + f^2(x_n) + g^2(x_n), \tag{38}$$

$$R_{n+1} = 2(1 + f(x_n))g(x_n)\xi_{n+1} + g^2(x_n)(\xi_{n+1}^2 - 1). \tag{39}$$

Remark 27. Under conditions of this section, $F_n, R_{n+1} \to 0$ a.s. as $n \to \infty$.

Let $a, \mu \in \mathbb{R}$, $\mu > 0$ and $\nu = \frac{\mu}{2}$. We define for $u > 0$

$$G(u) = \frac{u^{-\nu}}{a\nu}, \quad x > 0. \tag{40}$$

Assuming that $x_n \neq 0$, we apply the Taylor expansion with three terms to obtain for x_n^2 a.s.

$$\begin{aligned} G(x_{n+1}^2) &= G\Big(x_n^2 + x_n^2(F_n + R_{n+1})\Big) \\ &= G(x_n^2) - \frac{1}{a}|x_n|^{-2\nu}(F_n + R_{n+1}) + \frac{\nu+1}{2a}\eta_{n+1}^{-\nu-2}|x_n|^4(F_n + R_{n+1})^2, \end{aligned} \tag{41}$$

with

$$|\eta_{n+1} - x_n^2| \le x_n^2|F_n + R_{n+1}|. \tag{42}$$

Let

$$\kappa_{n+1} = \eta_{n+1}^{-(\nu+2)}x_n^{2(\nu+2)}.$$

We can rewrite (41) in the following form

$$\begin{aligned} G(x_{n+1}^2) = G(x_n^2) &- \frac{1}{a}|x_n|^{-\mu}(F_n + R_{n+1}) + \frac{\nu+1}{2a}|x_n|^{-\mu}(F_n + R_{n+1})^2 \\ &+ \frac{\nu+1}{2a}(\kappa_{n+1} - 1)|x_n|^{-\mu}(F_n + R_{n+1})^2 \\ &= G(x_n^2) + P_{n+1} + \rho_{n+1} + \tau_{n+1}, \end{aligned} \tag{43}$$

where

$$P_{n+1} = -\frac{2}{a}|x_n|^{-\mu}\left[f(x_n) - \frac{\mu+1}{2}g^2(x_n)\right] + Q_{n+1}, \tag{44}$$

$$\rho_{n+1} = -\frac{2}{a}|x_n|^{-\mu}g(x_n)\xi_{n+1}, \tag{45}$$

$$\tau_{n+1} = \frac{\mu+1}{a}|x_n|^{-\mu}g^2(x_n)(\xi_{n+1}^2 - 1), \tag{46}$$

$$\begin{aligned} Q_{n+1} = &-\frac{1}{a}|x_n|^{-\mu}\Big(f^2(x_n) + 2f(x_n)g(x_n)\xi_{n+1}\Big) \\ &+ \frac{\mu+2}{4a}|x_n|^{-\mu}\Big(F_n^2 + 2F_nR_{n+1} + (\kappa_{n+1} - 1)(F_n + R_{n+1})^2\Big) \\ &+ \frac{\mu+2}{4a}|x_n|^{-\mu}\Big(4f^2(x_n)g^2(x_n)\xi_{n+1}^2 + 8f(x_n)g^2(x_n)\xi_{n+1}^2 \\ &+ g^4(x_n)(\xi_{n+1}^2 - 1)^2 + 4g^3(x_n)(1 + f(x_n))(\xi_{n+1}^2 - 1)\xi_{n+1}\Big). \end{aligned} \tag{47}$$

Let

$$\mu = \begin{cases} \mu_f & \text{if} \quad \mu_f < \frac{\mu_g}{2}, \\ \mu_g - \mu_f & \text{if} \quad \frac{\mu_g}{2} \le \mu_f < \mu_g, \\ \frac{\mu_g}{2} & \text{if} \quad \mu_f > \mu_g. \end{cases} \tag{48}$$

Since ρ_n and τ_n defined by (45)-(46) are $\mathcal{F}_n$-martingale-differences,

$$M_n = \sum_{i=0}^{n-1} \rho_{i+1} \qquad \text{and} \qquad T_n = \sum_{i=0}^{n-1} \tau_{i+1} \tag{49}$$

are $\mathcal{F}_n$-martingales. After summation of (43), we arrive at

$$G(x_n^2) = G(x_0^2) + \sum_{i=0}^{n-1} P_{i+1} + M_n + T_n. \tag{50}$$

Remark 28. **(Outline of the proof of Theorems 25 and 26).** The decomposition of $G(x_{n+1}^2)$, equation (43), is constructed in a way that highlights the different types of convergence as $n \to \infty$ for different values of μ.

Firstly, the term P_{n+1} is made up from 3 parts. The last part, Q_{n+1}, is subdominant to the first two for all values of μ. When $\mu_f < \mu_g$ the limiting behavior of P_{n+1} is dominated by the function f, while when $\mu_f > \mu_g$, the function g determines the behavior.

When $\mu_f < \mu_g/2$ we set $\mu = \mu_f$ and show that both ρ_{n+1} and τ_{n+1} a.s. tend to zero as $n \to \infty$ and P_{n+1} tends to 1 as $n \to \infty$. This means that asymptotic behavior of $G(x_{n+1}^2) - G(x_n^2)$ is determined by P_{n+1} and the result can be obtained directly by Toeplitz lemma.

When $\mu_g > \mu_f \ge \mu_g/2$, the martingale-difference ρ_{n+1} no longer tends to zero if $\mu = \mu_f$. To get around this difficulty we set $\mu = \mu_g - \mu_f$ and show that in this situation both $P_{n+1} + \tau_{n+1}$ and $\mathbf{E}\left[\rho_{n+1}^2 \middle| \mathcal{F}_n\right]$ behave like $|x_n|^{2\mu_f - \mu_g}$ as $n \to \infty$. By means of a martingale convergence theorem, namely by Lemma 2, we compare the behavior of $G(x_n^2)$ with that of $\sum_{i=1}^{i=n} |x_i|^{2\mu_f - \mu_g}$ and apply Corollary 13.

When $\mu_f \ge \mu_g$, both ρ_{n+1} and τ_{n+1} decay slower than P_{n+1} for all values of μ. We set $\mu = \mu_g/2$ so that $\mathbf{E}\left[(\rho_{n+1} + \tau_{n+1})^2 \middle| \mathcal{F}_n\right] \to 1$ and $P_{n+1} \sim |x_n|^{\mu_g/2}$ as $n \to \infty$. This allows us to apply a consequence of the central limit theorem (see Lemma 33 below) and decomposition (50) to obtain the conclusion of Theorem 26.

The rest of this section we devote to verifying that Q_{n+1} is subdominant to the other terms in P_{n+1}.

Lemma 29. *There is some $K = K(\mu) > 0$ and $N = N(K, \omega)$ such that for all $n \ge N$ we have*

$$|\kappa_{n+1} - 1| \le K |F_n + R_{n+1}|. \tag{51}$$

In particular, a.s. $\kappa_n \to 1$ as $n \to \infty$.

Proof. From (42) we have

$$\left| \frac{\eta_{n+1}}{x_n^2} - 1 \right| \le |F_n + R_{n+1}|. \tag{52}$$

Since for any $\lambda \in \mathbb{R}$,

$$\lim_{y \to 1} \frac{y^\lambda - 1}{y - 1} = \lambda,$$

for any $K > |\lambda|$ we can find $\delta > 0$ such that

$$|y^\lambda - 1| \le K|y-1| \tag{53}$$

when $|y-1| < \delta$. Letting $\lambda = -(\mu/2+2)$, and using (52) and Remark 27, we can find $N(K,\omega)$ such that for all $n > N(K,\omega)$

$$|F_n + R_{n+1}| < \delta,$$

which, together with (53), implies (51). $\square$

Lemma 30. *Let μ be as defined in (48), and $s = \min\{\mu_f, \mu_g\}$. Then, for any $\varepsilon > 0$, a.s. $Q_{n+1} \sim O(x_n^{s-\varepsilon})$ as $n \to \infty$.*

Proof. By application of Corollary 23 for any $\varepsilon > 0$ we obtain that a.s.

$$\kappa_{n+1} - 1 = O\left(|x_n|^{\mu_f} + |x_n|^{\frac{\mu_g}{2}-\varepsilon}\right), \quad \text{as} \quad n \to \infty,$$

and then, from (47),

$$Q_{n+1} = O\left(|x_n|^{-\mu+2\mu_f} + |x_n|^{-\mu+\mu_f+\frac{\mu_g}{2}-\varepsilon} + |x_n|^{-\mu+\frac{3\mu_g}{2}-\varepsilon}\right). \tag{54}$$

Now the proof can be completed by direct substitution of different values of μ from (48) into (54). $\square$

6.3. **Proof of Theorem 25.**

6.3.1. *An auxiliary lemma.*

Lemma 31. *Let $a_f > 0$ and $\mu_f < \mu_g$. If we set $\mu = \mu_f$, and $a = 2a_f > 0$ in equations (44), (46) then*

$$\lim_{n\to\infty} \left[P_{n+1} + \tau_{n+1}\right] = 1, \quad \text{a.s.}.$$

The result follows from equations (31), (44), (46), Corollary 23 and Lemma 30.

6.3.2. *Proof of Theorem 25.* We consider two cases:

- (i) $\mu_f < \frac{1}{2}\mu_g$,
- (ii) $\frac{1}{2}\mu_g \le \mu_f < \mu_g$.

Proof of Theorem 25, case (i). We set $\mu = \mu_f$, and $a = 2a_f > 0$ in equations (44)-(47). By Corollary 23 for $\varepsilon < \frac{\mu_g}{2} - \mu_f$ we have a.s. as $n \to \infty$

$$\rho_{n+1} = O\left(|x_n|^{-\mu_f+\frac{\mu_g}{2}-\varepsilon}\right) \to 0.$$

Then from Lemma 31 we obtain that for $n \to \infty$

$$G(x_{n+1}^2) - G(x_n^2) \to 1, \quad \text{a.s.}$$

which together with Toeplitz Lemma and (40), implies that a.s. for $n \to \infty$

$$\frac{1}{na_f\mu_f}|x_n|^{-\mu_f} = \frac{G(x_n^2)}{n} \to 1.$$

$\square$

Proof of Theorem 25, case (ii). We set $\mu = \mu_g - \mu_f$, $a = \frac{2a_g}{a_f}$ in equations (44)-(47). We also denote

$$b := \frac{a_f^2}{a_g} > 0, \quad \lambda := 2\mu_f - \mu_g \geq 0.$$

Applying again Corollary 23 and Lemma 30 and reasoning in the usual way we get a.s. as $n \to \infty$

$$\frac{\mathbf{E}[\rho_{n+1}^2|\mathcal{F}_n]}{b|x_n|^\lambda} = \frac{g^2(x_n)|x_n|^{-2(\mu_g-\mu_f)}}{a_g|x_n|^{2\mu_f-\mu_g}} \to 1, \tag{55}$$

$$\frac{P_{n+1} + \tau_{n+1}}{b|x_n|^\lambda} \to 1. \tag{56}$$

Now we prove that a.s.

$$\lim_{n\to\infty} \sum_{i=1}^n |x_i|^\lambda = \infty. \tag{57}$$

Indeed, let $\Omega_1 = \{\omega : \lim_{n\to\infty} \sum_{i=1}^n |x_i|^\lambda < \infty\}$. Then from (55) and (56) we obtain that on Ω_1 we also have

$$\sum_{i=1}^\infty [P_{i+1} + \tau_{i+1}] < \infty \qquad \text{and} \qquad \sum_{i=1}^\infty \mathbf{E}[\rho_{i+1}^2|\mathcal{F}_i] < \infty.$$

The last relation implies that $\lim_{n\to\infty} M_n$ is a.s. finite on Ω_1 and then equation (50) implies that $\lim_{n\to\infty} G(x_n) < \infty$ a.s. on Ω_1. But since $x_n \to 0$ a.s., the probability of Ω_1 must be zero.

Relation (55) together with (57) implies that a.s.

$$\frac{\sum_{i=1}^n \mathbf{E}[\rho_{i+1}^2|\mathcal{F}_i]}{\sum_{i=1}^n b|x_i|^\lambda} \to 1,$$

which results in

$$\frac{M_n}{\sum_{i=1}^n b|x_i|^\lambda} \to 0. \tag{58}$$

On the other hand, relation (56) together with (57) implies that

$$\frac{\sum_{i=1}^n [P_{i+1} + \tau_{i+1}]}{\sum_{i=1}^n b|x_i|^\lambda} \to 1,$$

which together with (58) gives

$$\lim_{n\to\infty} \frac{G(x_n^2)}{\sum_{i=1}^n b|x_i|^\lambda} = 1.$$

After applying (40) and rearranging, we arrive at

$$|x_n|^\mu \sum_{i=1}^n |x_i|^\lambda \to \frac{1}{a_f\mu}. \tag{59}$$

Now we are going to apply Corollary 13 from Lemma 12. We set $\gamma = \lambda/\mu$ and $c = \left(\frac{1}{a_f\mu}\right)^\gamma$. We define $y_n := \sum_{i=1}^{n-1} |x_i|^\lambda$, so that $\Delta y_n = |x_n|^\lambda$ and $|x_n|^\mu = (\Delta y_n)^{1/\gamma}$. Now relation (59) takes the form

$$(\Delta y_n)^{1/\gamma} y_{n+1} \to \frac{1}{a_f\mu},$$

which, together with $\frac{y_{n+1}}{y_n} \to 1$, implies

$$(\Delta y_n)^{1/\gamma} y_n \to \frac{1}{a_f \mu},$$

or, equivalently,

$$y_n^\gamma \Delta y_n \to c. \tag{60}$$

By applying Corollary 13 we obtain

$$\frac{y_n}{n^{\frac{1}{1+\gamma}}} \to \big(c(1+\gamma)\big)^{\frac{1}{1+\gamma}}, \quad \text{or} \quad \frac{y_n^\gamma}{n^{\frac{\gamma}{1+\gamma}}} \to \big(c(1+\gamma)\big)^{\frac{\gamma}{1+\gamma}}.$$

The last limit together with (60) gives

$$\Delta y_n n^{\frac{\gamma}{1+\gamma}} = (y_n^\gamma \Delta y_n) \times \frac{n^{\frac{\gamma}{1+\gamma}}}{y_n^\gamma} \to c\big(c(1+\gamma)\big)^{-\frac{\gamma}{1+\gamma}}. \tag{61}$$

Substituting the values for Δy_n, γ and c in (61) gives the desired result. □

6.4. **Proof of Theorem 26.**

6.4.1. *Auxiliary lemmas.* The following lemma can be considered as a corollary of a version of strong law of large numbers for square-integrable martingales (see e.g. [20], page 519).

Lemma 32. *If M_n is a square-integrable martingale with the quadratic characteristic $\langle M_n \rangle$ and $\langle M_n \rangle \to \infty$, then for any $\gamma > 0$*

$$\lim_{n\to\infty} \frac{M_n}{(\langle M_n \rangle)^{1/2+\gamma}} = 0, \quad a.s.$$

Define $\Phi \in C(\mathbb{R};\mathbb{R})$ by

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy. \tag{62}$$

Assumption 2, due to Liapunov, gives rise to the following form of the Central Limit Theorem (see e.g. [6], page 362).

Lemma 33. *Let Assumption 2 hold. Then*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i > x\right] = 1 - \Phi(x), \quad \textit{for all } x \in \mathbb{R},$$

where Φ is given by (62).

The following result is then a simple adaptation of the argument presented on p.380–1 in [20].

Lemma 34. *Suppose that $\{\xi_n\}_{n\in\mathbb{N}}$ obeys Assumption 2. Then*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i = \infty, \quad \liminf_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i = -\infty, \quad a.s. \tag{63}$$

Proof. For $c > 0$ define the events

$$A_c = \{\omega : \limsup_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i > c\}, \quad A = \{\omega : \limsup_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i = \infty\}.$$

Then $A_c \downarrow A$ as $c \to \infty$. The events A_c are tail events; therefore, by independence of the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ and the Zero-One Law, it follows that

$$\mathbb{P}[A_c] > 0 \text{ for every } c > 0 \tag{64}$$

implies $\mathbb{P}[A_c] = 1$, and so $\mathbb{P}[A] = \lim_{c\to\infty} \mathbb{P}[A_c] = 1$. Therefore it suffices to prove (64) to establish the first part of (63).

Using (i) the fact that for any sequence of random variables $\{\eta_n\}_{n\in\mathbb{N}}$ we have

$$\{\omega : \limsup_{n\to\infty} \eta_n(\omega) > x\} \supseteq \{\omega : \eta_n(\omega) > x \text{ i.o.}\}, \quad \text{for all } x \in \mathbb{R},$$

(ii) the fact that $\mathbb{P}[B_n \text{ i.o.}] \geq \limsup_{n\to\infty} \mathbb{P}[B_n]$ for any sequence of events $\{B_n\}_{n\in\mathbb{N}}$, and then Lemma 33 in turn, we get

$$\begin{aligned} \mathbb{P}[A_c] &= \mathbb{P}\left[\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i > c\right] \geq \mathbb{P}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i > c \text{ i.o.}\right] \\ &\geq \limsup_{n\to\infty} \mathbb{P}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i > c\right] = \lim_{n\to\infty} \mathbb{P}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i > c\right] = 1 - \Phi(c), \end{aligned}$$

proving (64).

The second part follows from the first using the change $\xi_i \to -\xi_i$. □

Lemma 35. *Let $a = 2\sqrt{a_g}$ and $\mu = \mu_g/2$. Then*

$$\limsup_{t\to\infty} \frac{-(M_n + T_n)}{\sqrt{n}} = \infty, \quad \limsup_{t\to\infty} \frac{M_n + T_n}{\sqrt{n}} = \infty, \quad a.s. \tag{65}$$

Proof. We consider the case when $\mu_g < \mu_f$, and therefore we are under conditions of Corollary 2, b). Then, according to Remark 5, for any $\varepsilon > 0$

$$n^{-\frac{1}{\mu_g}-\varepsilon} \leq |x_n| \leq n^{-\frac{1}{\mu_g}+\varepsilon} \tag{66}$$

for all $n \geq N(\varepsilon, \omega)$. We choose $\varepsilon < \frac{1}{\mu_g}$ and in the following consider only $n \geq N(\varepsilon, \omega)$.

We define M_n and ρ_{n+1} as above and rearrange

$$\rho_{n+1} := -\xi_{n+1} + \bar{\rho}_{n+1},$$

where

$$\bar{\rho}_{n+1} = \left[1 - \frac{1}{\sqrt{a_g}} |x_n|^{-\mu_g/2} g(x_n)\right] \xi_{n+1} = o(|x_n|^r)\xi_{n+1}.$$

Let $\bar{M}_n = \sum_{i=1}^{n} \bar{\rho}_i$. We prove that

$$\lim_{n\to\infty} \frac{\bar{M}_n}{\sqrt{n}} = 0, \quad \lim_{n\to\infty} \frac{T_n}{\sqrt{n}} = 0,$$

which in conjunction with the conclusion of Lemma 34 ensures the desired result.

We want to estimate $\langle \bar{\rho}_n \rangle$. Since

$$|x_n|^r \leq n^{r\left(-\frac{1}{\mu_g}+\varepsilon\right)}$$

we have

$$\langle \bar{\rho}_n \rangle \leq |x_n|^{2r} \leq K n^{2r\left(-\frac{1}{\mu_g}+\varepsilon\right)} < n^{-\delta}$$

for some $\delta \in (0,1)$. Since, as $n \to \infty$,

$$\sum_{i=1}^{n} i^{-\delta} \sim \frac{1}{1-\delta} n^{1-\delta},$$

we have

$$\langle \bar{M}_n \rangle \le K \sum_{i=1}^{n} i^{-\delta} \sim K_1 n^{1-\delta}.$$

For fixed δ we choose $\gamma > 0$ such that $(1/2+\gamma)(1-\delta) < 1/2$. Indeed, it is possible for $\gamma < \frac{\delta}{2(1-\delta)}$. Then

$$\lim_{n\to\infty} \frac{\left(\langle \bar{M}_n \rangle\right)^{1/2+\gamma}}{\sqrt{n}} = 0, \quad \text{a.s.}$$

Hence by applying Lemma 32 we have

$$\lim_{n\to\infty} \frac{\bar{M}_n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\bar{M}_n}{\left(\langle \bar{M}_n \rangle\right)^{1/2+\gamma}} \lim_{n\to\infty} \frac{\left(\langle \bar{M}_n \rangle\right)^{1/2+\gamma}}{\sqrt{n}} = 0.$$

We define T_n and τ_{n+1} as before. Since

$$\tau_{n+1} = -\frac{\mu_g+2}{4}\sqrt{a_g}|x_n|^{\mu_g/2}\left[1+o(|x_n|^r)\right]^2 (\xi_{n+1}^2 - 1),$$

applying (66) with $\varepsilon < \frac{1}{2\mu_g}$ we estimate for $n \ge N(\varepsilon,\omega)$

$$\begin{aligned}\langle \tau_{n+1} \rangle &= \left(\frac{\mu_g+2}{4}\right)^2 a_g |x_n|^{\mu_g}\left[1+o(|x_n|^r)\right]^4 (\mathbf{E}\xi_{n+1}^4 - 1) \\ &\le K_2 |x_n|^{\mu_g}(\mathbf{E}\xi_{n+1}^4 - 1) \le K_3(\omega) n^{-1+\varepsilon\mu_g}(\mathbf{E}\xi_{n+1}^4 - 1).\end{aligned}$$

Then, as $n \to \infty$, because the fourth moments of ξ_n are uniformly bounded in n,

$$\langle T_n \rangle \le \sum_{i=1}^{n} K_3(\omega) i^{-1+\varepsilon\mu_g}(\mathbf{E}\xi_{i+1}^4 - 1) \le \sup_{i\in\mathbb{N}}\{\mathbf{E}\xi_i^4 - 1\} \times K_3(\omega)\sum_{i=1}^{n} i^{-1+\varepsilon\mu_g} \sim K_4 n^{\varepsilon\mu_g}.$$

Therefore, as $2\mu_g\varepsilon < 1$, we have $\langle T_n \rangle/\sqrt{n} \to 0$ as $n \to \infty$ a.s. Also, by the strong law of large numbers for martingales $T_n/\langle T_n \rangle \to 0$ as $n \to \infty$, a.s., so

$$\lim_{n\to\infty} \frac{T_n}{\sqrt{n}} = \lim_{n\to\infty} \frac{T_n}{\langle T_n \rangle} \lim_{n\to\infty} \frac{\langle T_n \rangle}{\sqrt{n}} = 0.$$

Applying Lemma 34 a.s. we get

$$\limsup_{n\to\infty} \frac{M_n + T_n}{\sqrt{n}} = \limsup_{n\to\infty} \frac{\sum_{i=1}^{n}(-\xi_i)}{\sqrt{n}} + \lim_{n\to\infty} \frac{\bar{M}_n}{\sqrt{n}} + \lim_{n\to\infty} \frac{T_n}{\sqrt{n}} = \infty,$$

as required. The proof of the 2nd part of (65) is similar. □

Lemma 36. *Let either $\mu_f > \mu_g$ or $\mu_f = \mu_g$ and $-2a_f < a_g$ hold. We set $\mu = \mu_g/2$, and $a = 2\sqrt{a_g} > 0$ in equation (44). Then*

$$\lim_{n\to\infty} \frac{P_{n+1}}{a(\mu+1)|x_n|^{\mu}} = S > 0, \tag{67}$$

where the number S is non-random.

Proof. From equation (31) and Lemma 30 we obtain that a.s. as $n \to \infty$

$$-\frac{2}{a^2(\mu+1)|x_n|^{2\mu}} f(x_n) \to L = \begin{cases} 0, & \mu_f > \mu_g, \\ \frac{a_f}{a_g(\mu_g+2)}, & \mu_f = \mu_g, \end{cases}$$

$$\frac{g^2(x_n)}{a^2|x_n|^{2\mu}} \to \frac{1}{4},$$

$$\frac{Q_{n+1}}{a(\mu+1)|x_n|^{\mu}} \to 0.$$

The above relations imply (67) with $S = L + \frac{1}{4}$. Note that $S > 0$ even if a_f is negative. Indeed, if this is the case and $\mu_f = \mu_g$

$$L = \frac{a_f}{a_g(\mu_g+2)} > \frac{-a_g}{2a_g(\mu_g+2)} = -\frac{1}{2(\mu_g+2)} > -\frac{1}{4},$$

since $-2a_f < a_g$. The lemma is proved. □

6.4.2. *Proof of Theorem 26.* We set $\mu = \mu_g/2$, $a = 2\sqrt{a_g}$.

Proof of (35). First we rearrange (50) in the following way

$$\sum_{i=0}^{n-1} P_{i+1} = G(x_n) - G(x_0) - M_n - T_n.$$

Due to Lemma 35,

$$\limsup_{n\to\infty} \frac{-(M_n+T_n)}{\sqrt{n}} = \infty,$$

and moreover, since $G(x_n) \to \infty$, we conclude that

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} P_{i+1}}{\sqrt{n}} = \infty. \tag{68}$$

By applying Lemma 36, we conclude that P_{n+1} is positive for big enough n. Thus $\sum_{i=0}^{n-1} P_{i+1}$ has a limit as $n \to \infty$, finite or infinite. Consequently, (68) implies that the limit is infinite. Together with Lemma 36 this implies that $\sum_{i=0}^{\infty} |x_i|^{\mu}$ can not be finite on a set of nonzero probability, i.e. $\sum_{i=0}^{\infty} |x_i|^{\mu} = \infty$ a.s. Therefore we can apply Toeplitz Lemma, or Lemma 11, and obtain that

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} P_{i+1}}{a(\mu+1)\sum_{i=0}^{n-1} |x_i|^{\mu}} = \lim_{n\to\infty} \frac{P_{n+1}}{a(\mu+1)|x_n|^{\mu}} = S > 0.$$

Combining this with (68) gives

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} |x_i|^{\mu}}{\sqrt{n}} = \infty.$$

Since

$$\sqrt{n} > \frac{1}{2}\sum_{i=1}^{n} i^{-\frac{1}{2}},$$

we can estimate

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} |x_i|^{\mu}}{\frac{1}{2}\sum_{i=1}^{n-1} i^{-\frac{1}{2}}} = \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} |x_i|^{\mu}}{\frac{1}{2}\sum_{i=1}^{n} i^{-\frac{1}{2}}} \ge \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} |x_i|^{\mu}}{\sqrt{n}} = \infty.$$

Applying Lemma 11 again, from the last limit we conclude that

$$\limsup_{n\to\infty} \sqrt{|x_n|^{\mu_g} n} = \limsup_{n\to\infty} |x_n|^{\mu}\sqrt{n} = \infty.$$

□

Proof of (36). In the previous part we proved that $\sum_{i=0}^{\infty} P_{i+1} = \infty$, therefore

$$\liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} P_{i+1}}{\sqrt{n}} \geq 0.$$

After dividing both parts of the decomposition (50) by $\sqrt{n}$, taking the limsup of both parts and applying Lemma 35 we obtain

$$\limsup_{n\to\infty} \frac{G(x_n)}{\sqrt{n}} = \limsup_{n\to\infty} \left[\frac{G(x_0)}{\sqrt{n}} + \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} P_{i+1} + \frac{M_n + T_n}{\sqrt{n}} \right]$$
$$\geq \limsup_{n\to\infty} \frac{M_n + T_n}{\sqrt{n}} = \infty.$$

Therefore,

$$\infty = \limsup_{n\to\infty} \frac{G(x_n)}{\sqrt{n}} = \frac{1}{a\mu} \limsup_{n\to\infty} \frac{|x_n|^{-\mu}}{\sqrt{n}},$$

or

$$\liminf_{n\to\infty} \sqrt{n} |x_n|^{\mu_g} = 0.$$

The theorem is proved. □

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