Dynamics
of
Interval Maps with Hysteresis

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Abstract

We consider a special case of multistate maps, maps with hysteresis. The map under consideration is a collection of two continuous, monotone real-valued functions with overlapping domains of definition. At each step we determine the function to apply using the following rule: if the current iterate of the initial point is in the domain of definition of the function we applied last then we apply this function again, otherwise the second function is applied.

We study two different aspects of such maps: topological and combinatorial. The topological object of study is the global attractor (the limit image of the whole space under the map). We review general properties of the global attractor of a continuous map. However, maps with hysteresis are not continuous. To fix this, we consider two approaches to the construction of a continuous map with hysteresis. The first approach extends the map itself, converting it to an upper semicontinuous set-valued map, while the second one redefines the space, on which the map acts.

We consider a parameterized family of maps with hysteresis. After establishing some results on continuity of the global attractor as a function of the parameter, a more detailed analysis of a special case of general maps with hysteresis, a piecewise linear map with hysteresis, is presented. In two different cases, when there are periodic points and when there are none, we describe the global attractor, its continuity properties and points where discontinuities occur.

Combinatorial aspects of the maps are explored by means of kneading sequences and kneading invariants. We prove one-to-one correspondence between possible kneading invariants and equivalence classes of maps with hysteresis, where by equivalence we understand topological conjugacy.

Contents

Chapter 1

Introduction

Given a metric space Y and an index set S, which may be discrete or continuous, define for each $s \in S$ a subset of Y, U_s. By a *multistate map* we mean a discrete time dynamical system defined on

$$X = \bigcup_{s \in S} U_s \times \{s\} \subset Y \times S. \tag{1.1}$$

We call the elements of Y *observables*, while elements of S are *states*. Given an observable x_n and state s_n, we generate a new observable x_{n+1} by the transformation

$$x_{n+1} = F(x_n,\, s_n).$$

In turn, having determined the new observable x_{n+1} we generate a new state s_{n+1} by

$$s_{n+1} = G(x_{n+1},\, s_n).$$

In this work we study a special case of multistate maps, *interval maps with hysteresis*. Here the index set $S = \{0, 1\}$ and the metric space $Y = \mathbb{R}^1$. Functions $F(\cdot, 0) = f_0$ and $F(\cdot, 1) = f_1$ are continuous nondecreasing functions defined on intervals $[a, \beta]$ and $[\alpha, b]$ respectively, where $\beta \geq \alpha$,

$$f_0(x) \geq x, \qquad f_1(x) \leq x$$

and

$$f_0(\beta) = b \quad \text{and} \quad f_1(\alpha) = a.$$

Thus, the space of Eq. (1.1) reduces to

$$X_h = \Big([a, \beta] \times \{0\}\Big) \cup \Big([\alpha, b] \times \{1\}\Big) \tag{1.2}$$

Throughout the work, a point $\mathbf{x} \in X_h$ will mean the whole pair (x, s). Sometimes we use functions $\mathrm{Obs}(\mathbf{x})$ and $\mathrm{St}(\mathbf{x})$ to refer to observable x and state s, respectively.

The topology on the space X_h is induced by the standard $\mathbb{R}$ topology, *i.e.* $U \subset X_h$ is open if and only if

$$U = \Big((U_0 \cap [a, \beta]) \times \{0\}\Big) \cup \Big((U_1 \cap [\alpha, b]) \times \{1\}\Big),$$

where U_0 and U_1 are open subsets of real line. In the similar way we define the measure on X_h, induced by Lebesgue measure on $\mathbb{R}$,

$$\mu(U) = \mu\left(U_0 \cap [a, \beta]\right) + \mu\left(U_1 \cap [\alpha, b]\right),$$

the partial ordering of X_h (we compare only points of the same state) and the distance ρ between two points of the same state. We extend the definition of the metric ρ on X_h to points of any state by setting $\rho(\mathbf{x}, \mathbf{y}) = P$, $\mathrm{St}(\mathbf{x}) \neq \mathrm{St}(\mathbf{y})$, where constant P is sufficiently large to ensure triangle inequality. With this metric X_h becomes a compact metric space.

The mapping itself is defined on X_h as follows: $f\left(x_i, s_i\right) = (x_{i+1}, s_{i+1})$, where

$$x_{i+1} = f_{s_i}(x_i) \quad \text{and} \quad s_{i+1} = \begin{cases} 0 \text{ if } x_{i+1} \in [a, \alpha) \\ 1 \text{ if } x_{i+1} \in (\beta, b] \\ s_i \text{ otherwise} \end{cases}$$

with an initial point $(x_0, s_0) \in X_h$

As one can see, the periods of action of the two functions alternate and each function, f_0 and f_1, is applied as long as possible. The state switches when the observable leaves the domain of definition of the corresponding function. An example of a map with hysteresis and a typical trajectory are shown on Fig. B.1.

When $\beta = \alpha$ the map f reduces to a single-valued function with one discontinuity, a *Lorenz-type map*. This type of map is thoroughly studied in the literature [1, 2].

In our work we develop a theory for general maps with hysteresis and examine a special case, piecewise linear map with hysteresis (PLMH), in detail. The PLMH is given by

$$f_0(x) = \gamma_0 x, \quad f_1(x) = \gamma_1 x,$$

where $\gamma_0 > 1 > \gamma_1$, $\beta > \alpha$ and $a = \gamma_1 \alpha$, $b = \gamma_0 \beta$. An example of the PLMH is shown on Fig. B.2.

The dynamics of a piecewise linear map strongly depends on whether or not there are integers k and l such that $\gamma_0^k \gamma_1^l = 1$. In the former case all points are eventually periodic and in the latter there are no periodic points at all (the proof will be given in Lemma 11).

The global attractor (a definition will be given below) of the PLMH, obtained with the aid of computer simulation, has a very interesting structure. A typical example is given in Fig. B.3. The C programme which produced this picture is included in Appendix A, but we describe its structure here. One of the parameters of PLMH is being varied, e.g. Fig. B.3 is produced by varying the second threshold value, β, and keeping γ_0, γ_1 and α fixed. For every value of the parameter the programme takes a large number of points from X_h (points are distributed uniformly in a subinterval of X_h), performs a number of preliminary iterates to stabilize the process and then gives the next iterates of these points as an output. Thus for each value of the parameter, the output is a set of iterates of some points, which roughly corresponds to the ω-limit set of these points (all definitions are given in later Chapters).

The set shown in Fig. B.3 is an approximation to the *global attractor*[3], defined by

$$L = \bigcap_{i=0}^{\infty} f^i(X_h). \tag{1.3}$$

Although the set defined by Eq. (1.3) frequently contains no information about classical (continuous) discrete dynamical systems it plays a very important role in the case of maps with hysteresis. After presenting various definitions and recounting some useful facts in Chapter 2 we thoroughly study the set L in Chapter 3. The set L is an attractor according to various definitions and, when considering a parametric family of maps, its graph vs the parameter is upper semicontinuous.

However these useful properties are established assuming the continuity of the map f. The lack of continuity in the general map with hysteresis may be removed by extending the definition of the map. Two possible extensions are presented in Chapter 4. The first variant extends the map itself while the second one redefines the space X_h. Each definition has its own advantages, for example, the first is convenient in considerations of the bifurcation diagram (the graph of the set L vs some parameter) and the second one is used in the theory of kneading invariants.

In Chapter 5, while studying PLMH, we prove theorems about the two important threshold points $(\alpha, 1)$ and $(\beta, 0)$ and their images and preimages. If the map f has no periodic points, the preimages of α and β turn to be everywhere dense. This property is very useful in kneading theory and corresponds to topological expansiveness in the theory of Lorenz maps. Then we prove that the set L is the union of omega-limit sets of $(\alpha/\gamma_1, 1)$ and $(\beta/\gamma_0, 0)$ if these points are not mapped on to one-another. Now one of the main results of the chapter, that L is a non-wandering set of the map f, is an easy corollary of the above. Furthermore, we are able to prove that in this case the set L is the omega-limit set of any point $\mathbf{x} \in X_h$. These results allow us to reveal additional properties of the set L, as function of a parameter. The graph of L turns out to be lower semicontinuous at certain points, in addition to the upper semicontinuity proved in Chapter 4. We also study the graph of L when

$$\gamma_0^k \gamma_1^l = 1 \tag{1.4}$$

for some integers k and l. The boundary of the graph is shown to be contained in a simple set and this allows us to prove some additional results on continuity.

In Chapter 6 we return to the general case of a map with a hysteresis. Under assumption that the map is topologically expansive we develop a theory of kneading sequences. Then we define the kneading invariant to be the set of kneading sequences of the points a, b, α and β and state the main theorem of that chapter: there is a set of inequalities such that a set of four sequences is the kneading invariant of a map with hysteresis if and only if the inequalities are satisfied.

Finally, we give an overview of the work and projected research in the Summary.

Chapter 2

Basic definitions and notation

2.1 Basic definitions

Notation and definitions, used throughout the work, are included here for ready reference. Less known and more specific definitions will appear in the course of the report.

By X we denote an arbitrary compact metric space and X_h is the space defined by Eq. (1.2). For a set $A \subset X$, $B_\epsilon(A)$ is the open set of points within distance ϵ of A. The boundary of a set A is the set

$$\partial(A) = \{x \colon \forall \epsilon \; B_\epsilon(x) \cap A \neq \emptyset \text{ and } B_\epsilon(x) \setminus A \neq \emptyset\}\,.$$

By $\mathrm{Int}(A)$ and $\overline{A}$ we denote interior and closure of set A respectively,

$$\mathrm{Int}(A) = A \setminus \partial(A), \qquad \overline{A} = A \cup \partial(A).$$

In order to compensate for the discontinuity of a map with hysteresis we will be considering its set-valued extension (see Chapter 4). The following five definitions, although formulated for single-valued maps, remain unchanged in the set-valued case.

Let f be a (single or set-valued) map. The following standard notation will be used in our study: the image of a set A under f is

$$f(A) = \{f(x) \colon x \in A\} = \bigcup_{x \in A} f(x).$$

Iterations of the map f are defined by induction

$$f^{k+1}(A) = \bigcup_{x \in f^k(A)} f(x).$$

Given a set A we define the set of its images by

$$\mathrm{Img}(A) = \bigcup_{i=0}^{\infty} f^i(A).$$

Definition 1 *A set A is called* forward invariant *if $f(A) \subset A$. It is called* invariant *if $f(A) = A$.*

In other words, a set A is invariant if and only if it is forward invariant and weakly backward invariant (for any $x \in A$ there is at least one $y \in A$ such that $x \in f(y)$, see also [4]).

Definition 2 *A point x is a* periodic point *for f if $x \in f^n(x)$ for some $n > 0$. A point is called* eventually periodic *if $f^k(x)$ contains a periodic point for some k.*

Definition 3 *The* ω-limit set *of a set U is the set*

$$\omega(U) = \left\{ x \in X \colon \exists \left\{n_i\right\}_{i=0}^{\infty},\ \exists \left\{y_i\right\}_{i=0}^{\infty} \subset U,\ \exists x_i \in f^{n_i}\left(y_i\right)\ \Big(x_i \to x\Big) \right\}.$$

Definition 4 *A point $x \in X$ is called* non-wandering *if for any open $U \subset X$, $x \in U$, there is an integer k such that $f^k(U) \cap U \neq \emptyset$. The set Ω of all non-wandering points is called the* non-wandering set*.*

Definition 5 *A point x^{-k} is said to be a k-*preimage *of x under a map f if $x \in f^k\left(x^{-k}\right)$.*

To introduce notions of continuity for set-valued maps we need a metric on the space of closed subsets of X.

Definition 6 *The distance from a closed set A to a closed set B is*

$$\rho_*(A, B) = \sup_{a \in A} \rho(a, B),$$

where $\rho(a, B) = \inf_{b \in B} \rho(a, b)$. The Hausdorff metric *$\rho$ is then defined by*

$$\rho(A, B) = \max\left\{\rho_*(A, B),\ \rho_*(B, A)\right\}.$$

Now let X and Y be compact metric spaces.

Definition 7 *A set-valued function* $f : X \to C(Y)$*, where* $C(Y) = \{F \subset Y : F \text{ is closed}\}$*, is* upper semicontinuous *at* x_0 *if*

$$\lim_{x \to x_0} \rho_*(f(x), f(x_0)) = 0.$$

Definition 8 *A set-valued function* $f : X \to C(Y)$ *is* lower semicontinuous *at* x_0 *if*

$$\lim_{x \to x_0} \rho_*(f(x_0), f(x)) = 0.$$

There are also alternative (equivalent) definitions we will make use of. For upper semicontinuity it is formulated in Theorem 1 below. For lower semicontinuity it is the following: a set-valued function is lower semicontinuous if for any point $y \in f(x_0)$ and any sequence $\{x_i\} \to x_0$ there is sequence $\{y_i\}$, $y_i \in f(x_i)$ such that $y_i \to y_0$.

Definition 9 *We say that function* $F\colon X \to C(Y)$ *is* continuous *at* x_0 *if it is upper and lower semicontinuous at* x_0*.*

Definition 10 *Let* μ *be a measyre on the space* Y*. A set-valued function* $f : X \to C(Y)$ *is* measure-continuous *at* x_0 *if*

$$\lim_{x \to x_0} \mu\left(f(x_0) \triangle f(x)\right) = 0,$$

where $A \triangle B = A \setminus B \cup B \setminus A$*.*

Definition 11 *The* graph *of a set valued function* $f : X \to C(Y)$ *is a subset of* $X \times Y$*:*

$$\mathrm{Graph}(f) = \{(x, y) \in X \times Y \colon y \in f(x)\}$$

In our study of the set L as a function of a parameter λ we will need a notion of convergence of set-valued functions.

Definition 12 *Let* f_n *be a sequence of set-valued maps. We say that it is* weakly upper convergent *to a map* f *if for any subsequence* $\{n'\}$

$$\forall x_{n'}\ \forall y_{n'} \in f_{n'}(x_{n'})\ \Big((x_{n'} \to x) \wedge (y_{n'} \to y) \Rightarrow y \in f(x)\Big)$$

Loosely speaking, if there is a sequence $\{(x_{n'}, y_{n'})\}$ in the graphs of the functions f_n which converges to a point (x, y) then $y \in f(x)$. Note, that this notion differs from upper graphical convergence [5]: in our case the graph of f may be *bigger* then the upper limit of graphs of f_n. We introduce this difference in order to ensure that this property is inherited by iterated functions f_n^k, see Lemma 1.

Definition 13 *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of set-valued maps. We say, that it is* weakly upper continuous *at a point λ_0 if for any sequence $\lambda_n \to \lambda_0$ the sequence of functions f_{λ_n} is weakly upper convergent to the function f_{λ_0}.*

We will also make use of the lower variant of convergence of maps. Again, our definition of *weak lower convergence* differs from lower graphical convergence [5].

Definition 14 *Let f_n be a sequence of set-valued maps. We say that it is* weakly lower convergent *to a function f if for any point $y_0 \in f(x_0)$ and any sequence $\{x_n\} \to x_0$*

$$\exists y_n \in f_n(x_n) \ \ (y_n \to y_0).$$

A family $\{f_\lambda\}_{\lambda\in\Lambda}$ of set-valued maps is said to be weakly lower continuous *at a point λ_0 if for any sequence $\lambda_n \to \lambda_0$, the sequence of functions f_{λ_n} is weakly lower convergent to the function f_{λ_0}.*

Definition 15 *A family $\{f_\lambda\}_{\lambda\in\Lambda}$ of set-valued maps is* weakly continuous *if it is weakly upper and lower continuous.*

2.2 Some useful facts

Theorem 1 *A set-valued map is upper semicontinuous if and only if its graph is closed.*

This theorem is well-known and we refer, for example, to [5] for the proof.

Lemma 1 *Let a family $\{f_\lambda\}_{\lambda\in\Lambda}$ of set-valued maps be weakly upper continuous at a point λ_0. Then for every k the family $\left\{f_\lambda^k\right\}_{\lambda\in\Lambda}$ is weakly upper continuous at λ_0.*

Proof. We prove this lemma by induction. We assume that statement is true for $k-1$, *i.e.* $\left\{f_\lambda^{k-1}\right\}_{\lambda\in\Lambda}$ is weakly upper continuous at λ_0. We want to prove that if $x_n \to x$ and there are $y_n \in f_{\lambda_n}^k(x_n)$ such that $y_n \to y$ then $y \in f_{\lambda_0}^k(x)$.

Each y_n has a preimage z_n,

$$z_n \in f_{\lambda_n}^{k-1}(x_n), \qquad y_n \in f_{\lambda_n}(z_n).$$

We choose a convergent subsequence from $\{z_n\}$:

$$\exists x_{n'} \text{ and } \exists z_{n'} \in f_{\lambda_{n'}}^{k-1}(x_{n'})\Big(z_{n'} \to z\Big).$$

The subsequence $\{x_{n'}\}$ converges to x and the assumption that $\left\{f_\lambda^{k-1}\right\}_{\lambda\in\Lambda}$ is continuous implies that $z \in f_{\lambda_0}^{k-1}(x)$. On the other hand, we have

$$z_{n'} \to z, \quad y_{n'} \in f_{\lambda_{n'}}(z_{n'}) \text{ and } y_{n'} \to y.$$

Since f_λ are continuous at λ_0 this means that $y \in f_{\lambda_0}(z)$. Together with the previous observation we get that $y \in f_{\lambda_0}^k(x)$. Q.E.D.

Corollary 1 *If a set-valued map f is upper semicontinuous then its k-th iterate f^k is also upper semicontinuous for any k.*

Indeed, if we take the family $\{f_\lambda\}_{\lambda\in\Lambda}$ with $f_\lambda \equiv f$ for every λ, the definition of weak upper continuity of the family reduces to the definition of upper semicontinuity of the map f and we can apply Lemma 1 to obtain the result.

A lemma, similar to Lemma 1, is true about weak lower continuity

Lemma 2 *Let a family $\{f_\lambda\}_{\lambda\in\Lambda}$ of set-valued maps be weakly lower continuous at a point λ_0. Then for every k the family $\left\{f_\lambda^k\right\}_{\lambda\in\Lambda}$ is weakly lower continuous at λ_0.*

Proof. Again we use induction for our proof. Let the statement be true for $k-1$. We want to prove that for any point $y_0 \in f_{\lambda_0}^k(x_0)$ and any sequences $\{\lambda_n\} \to \lambda_0$ and $\{x_n\} \to x_0$ there is a sequence $\{y_n\} \to y_0$, $y_n \in f_{\lambda_n}^k(x_n)$.

Let the point z_0 be such that

$$z_0 \in f_{\lambda_0}^{k-1}(x_0) \quad \text{and} \quad y_0 \in f_{\lambda_0}(z_0).$$

By the definition of weak lower continuity and assumption of induction there is a sequence $\{z_n\} \to z_0$, $z_n \in f_{\lambda_n}^{k-1}(x_n)$. We apply the definition of weak lower continuity once more to get a sequence $\{y_n\}$, $y_n \in f_{\lambda_n}(z_n)$ such that $y_n \to y_0$. It is clear that the sequence $\{y_n\}$ is the one we need. This observation finishes the proof.

Lemma 3 *If a set-valued map $f\colon X \to C(Y)$ is measure-continuous at a point x_0 then there is a set $N \subset f(x_0)$, $\mu(N) = 0$, such that the set-valued map $\widetilde{f}(x) = f(x) \setminus N$ is lower semicontinuous at x_0.*

Proof. We put

$$N = \{y \in f(x_0)\colon \exists \text{ open } U_y \ni y,\ \mu(U_y \cap f(x_0)) = 0\}.$$

The set N has measure zero. Indeed, N admits the representation

$$N = \bigcup_{y \in N} U_y \cap f(x_0),$$

where $U_y \cap f(x_0)$ has measure zero and open sets U_y are chosen from a countable base for the topology. The set of different possible U_y is at most countable, therefore the union above consists of at most a countable number of distinct sets and we can use σ-additivity of the measure to conclude that $\mu(N) = 0$.

We are going to prove that for any point $y \in f(x_0) \setminus N$ and any sequence $\{x_i\} \to x_0$ there is a sequence $\{y_i\} \to y$, $y_i \in f\{x_i\}$. Assume the contrary, there is a sequence $\{x_i\} \to x_0$ and an open neighbourhood U of $y \in f(x_0) \setminus N$ such that $U \cap f(x_i) = \emptyset$ for any i. Then

$$U \subset f(x_i) \setminus f(x_0)$$

and the measure $\mu(f(x_0) \bigtriangleup f(x)) \geq \mu(U \cap (f(x_0) \setminus N)) > 0$ for any i. This is in contradiction to the measure-continuity of the function f at the point x_0. Q.E.D.

Chapter 3

Global attractor and its properties

Let f be an upper semicontinuous set-valued map, $f\colon X \to C(X)$, on a compact metric space X.

We define the *global attractor* [3] of the space X under the map f by

$$L = \lim_{n\to\infty} f^n(X) = \bigcap_{i=0}^{\infty} f^n(X).$$

The set L is non-empty, closed and invariant: $f(L) = L$. Indeed, $f(X) \subset X$, therefore

$$f^{n+1}(X) = f^n(f(X)) \subset f^n(X).$$

The sets $f^n(X)$ are closed for every n and the global attractor

$$L = \bigcap_{i=0}^{\infty} f^n(X)$$

is also closed. This representation also implies that L is non-empty. To prove invariance we need some additional reasoning.

The inclusion $f(L) \subset L$ is trivial. To prove that $L \subset f(L)$ we assume the contrary: there exists a point $x \in L$ which does not have a preimage in L. In other words, $f^{-1}(x) \bigcap L = \emptyset$, where $f^{-1}(x)$ is the set of all 1-preimages of the point x. $f^{-1}(x)$ is a closed set, because f is upper semicontinuous. Then the open set $V = X \setminus f^{-1}(x) \supset L$ is such that $V \bigcap f^{-1}(x) = \emptyset$. Using the definition of L we infer that $f^n(X) \subset V$ for some n and, therefore, $f^{n+1}(X) \not\ni x$, which contradicts the hypothesis $x \in L$.

Definition 16 *A set A is called an* attractor *if there is an open set U, $U \supset A$, such that the ω-limit set of U is A.*

Lemma 4 *The global attractor L is an attractor according to Definition 16 and $\omega(X) = L$.*

Proof. First we note that $\omega(L) = L$. Indeed, for any point $x \in L$ we can find a preimage $y_1 \in L$ of x, then preimage $y_2 \in L$ of y_1 *et cetera* which eventually forms the sequence used in Definition 3.

Then, as $\omega(X) \supset \omega(L) = L$ we have to prove that any x satisfying $x \in \omega(X)$ is in the set L. The definition of the ω-limit set provides sequences of points $\{y_i\}_{i=0}^{\infty}$ and of iterations $\{n_i\}_{i=0}^{\infty}$ such that

$$\exists x_i \in f^{n_i}(y_i) \Big(x_i \to x \Big).$$

Since X is compact the sequence $\{z_i\}_{i=k}^{\infty}$,

$$z_i \in f^{n_i - k}(y_i), \qquad x_i \in f^k(z_i),$$

has a condensation point, x^{-k}, for any k. Without loss of generality we assume that $\{z_i\}$ itself converges to the point x^{-k}. Finally, we have

$$z_i \to x^{-k}, \; x_i \in f^k(z_i), \; x_i \to x.$$

We use upper continuity of the function f^k (Corollary 1) to infer that $x \in f^k\left(x^{-k}\right)$. Thus, $x \in f^k(X)$ for any k and, therefore, $x \in L$.

Now we take $U = X$ in Definition 16 to finish the proof.

Another possible definition of an attractor involves an open set U which is mapped into itself [6]:

Definition 17 *A set $A \subset X$ is called an* attractor *if for any $\epsilon > 0$ there is an open set U of positive Lebesgue measure in the ϵ-neighbourhood of A such that $A \subset U$, $f(U) \subset U$ and $x \in U$ implies $\omega(x) \in A$.*

Lemma 5 *The global attractor L is an attractor according to Definition 17.*

Proof. In order to show that L satisfies the definition we have to find a neighbourhood U of L such that $f(U) \subset U$; the second condition is satisfied since $\omega(x) \subset \omega(X) = L$ for any $x \in X$.

Step 1. For any $\epsilon > 0$ and set U satisfying

$$L \subset U \subset B_\epsilon(L), \qquad f(U) \subset U$$

there is a $\delta_0 > 0$ such that

$$f^k\left(B_\delta(U)\right) \subset B_\epsilon(L)$$

for any k and $\delta < \delta_0$.

Assume the contrary: there is a decreasing sequence $\delta_n \to 0$ and sequences $\{k_n\}$ and $\{x_n\}$ such that

$$x_n \in X \setminus B_\epsilon(L) \quad \text{and} \quad x_n \in f^{k_n}\left(B_{\delta_n}(U)\right)$$

Since $X \setminus B_\epsilon(L)$ is compact we can assume $x_n \to x_0 \notin B_\epsilon(L)$ without loss of generality. There are two cases to consider:

- $\{k_n\}$ is unbounded. Then, according to Definition 3, x_0 belongs to $\omega(X) = L$. But $L \subset U \subset B_\epsilon(L)$, which is a contradiction.

- $\{k_n\}$ is bounded. Then there is a number k_0 which is repeated in $\{k_n\}$ infinitely many times. We assume that $k_n \equiv k_0$ for any n without loss of generality. Then $x_n \in f^{k_0}(y_n)$, where $y_n \in B_{\delta_n}(U)$. As $\delta_n \to 0$ the sequence $\{y_n\}$ converges to the set U and, therefore, has a point $y \in U$ among its limit points. Due to the continuity of f^{k_0} and the property $f(U) \subset U$ we have

$$y_n \to y,\ x_n \in f^{k_0}\left(y_n\right),\ x_n \to x_0 \ \Rightarrow\ x_0 \in f^{k_0}(y) \subset U \subset B_\epsilon(L),$$

  which is a contradiction.

Step 2. For any $\epsilon > 0$ there exists an open set $U \supset L$ such that

$$U \subset B_\epsilon(L), \qquad f(U) \subset U$$

We define sets $U_n = f\left(\widetilde{U}_{n-1}\right)$, where $\widetilde{U}_n = B_\rho(U_n)$, with ρ, depending on n, being such that

$$f^k\left(B_\rho\left(U_n\right)\right) \subset B_\epsilon(L), \tag{3.1}$$

for any k. The initial set is $U_0 = L$. To prove that condition (3.1) is possible to satisfy we use induction. For $n = 0$ it is possible due to Step 1. Assume the statement is true up to the $n-1$-th step. We consider the set $W_n = \bigcup_{i=0}^{\infty} f^i\left(\bigcup_{j=0}^{n-1}\widetilde{U}_j\right)$, which is invariant under f and is contained in $B_\epsilon(L)$ by our assumption. Therefore W_n satisfies the conditions of Step 1 and we choose the next ρ to obtain $f^k\left(B_\rho\left(W_n\right)\right) \subset B_\epsilon(L)$. Then $\widetilde{U}_n = B_\rho\left(U_n\right) \subset B_\rho\left(W_n\right)$ and condition (3.1) is satisfied.

Finally we put $U = \bigcup_{k=0}^{\infty}\widetilde{U}_k$ which finishes the proof of Step 2: U is forward invariant, because $f\left(\widetilde{U}_k\right) \subset \widetilde{U}_{k+1}$, is contained in $B_\epsilon(L)$ and its measure is

$$\mu(U) \geq \mu\left(\widetilde{U}_0\right) = \mu\left(B_\rho(L)\right) > 0.$$

The lemma is proven.

Lemma 6 *Let $f\colon X \to C(X)$ be an upper semicontinuous map. Then the nonwandering set Ω is contained in the global attractor L.*

Proof. Let $U_n = B_{\epsilon_n}(X)$ be a sequence of open neighbourhoods of a point $x \in \Omega$ with $\epsilon_n \to 0$. Let $\{k_n\}$ be a positive sequence such that $k = k_n$ is the minimal number to satisfy $f^k\left(U_n\right) \cap U_n \neq \emptyset$. We consider two cases.

Sequence $\{k_n\}$ is bounded. Then there is a subsequence of indices, $\{n'\}$ such that $k_{n'} = k$. From the continuity of f we imply that the point x is k-periodic, $x \in f^k(x)$, and, therefore, $x \in L$.

Sequence $\{k_n\}$ is not bounded. Then for any i there is a subsequence $\{n'\}$ such that $k_{n'} > i$, therefore any point from $f^{k_{n'}}\left(U_{n'}\right) \cap U_{n'}$ has i-preimages. These preimages have an accumulation point, x_i, and by continuity $x \in f^i\left(x_i\right)$. Q.E.D.

Now let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of set-valued maps weakly upper continuous at a point λ_0 (for more general results, see [7]).

Theorem 2 *The function $L(\lambda)$ is upper semicontinuous at λ_0.*

Proof.

It is sufficient to prove, that the graph of $L(\lambda)$ is closed in the space $\mathbb{R}\times X$ (see Theorem 1). Let $\{(\lambda_i, x_i)\}_{i=1}^{\infty}$ be a convergent sequence with $x_i \in L(\lambda_i)$ and $(\lambda_i, x_i) \to (\lambda_0, x_0)$. We want to prove that $x_0 \in L(\lambda_0)$, thus we have to find a k-preimage of x_0 under f_{λ_0} for any k.

Let x_i^{-k} be a k-preimage of the point x_i under f_{λ_i}. Since X is compact, we assume without loss of generality that sequence converges $x_i^{-k} \to x^{-k}$. Then we have

$$x_i^{-k} \to x^{-k}, \quad x_i \in f_{\lambda_i}^k\left(x_i^{-k}\right), \quad x_i \to x_0.$$

Since $\{f_\lambda\}_{\lambda\in\Lambda}$ is assumed to be weakly upper continuous we apply Lemma 1 to infer that f_λ^k is also continuous and, therefore, $x_0 \in f^k\left(x^{-k}\right)$. Thus x^{-k} is k-preimage of the point x_0. Q.E.D.

Chapter 4

Continuous maps with hysteresis and their properties

4.1 Definition of continuous map with hysteresis

A map with hysteresis, as defined in Chapter 1, is not continuous.

Notation. If the points $\left(f_1^{-1}(\alpha), 1\right)$ and $\left(f_0^{-1}(\beta), 0\right)$ belong to the space X_h we call them *discontinuity points* and denote them by α^{-1} and β^{-1} respectively.

At the point α^{-1}, the map f is continuous from the right only:

$$f(\mathrm{Obs}(\mathbf{x}), 1) = \begin{cases} (f_1(\mathrm{Obs}(\mathbf{x})), 1) \text{ if } \mathbf{x} \geq \alpha^{-1} \\ (f_1(\mathrm{Obs}(\mathbf{x})), 0) \text{ if } \mathbf{x} < \alpha^{-1} \end{cases},$$

where $\mathrm{St}(\mathbf{x}) = 1$ and therefore comparison of $\mathbf{x}$ with α^{-1} is legitimate. The situation is the same with the point β^{-1}, but here the map f is continuous from the left. In order to make use of facts derived in the previous chapter we have to redefine f in such a way that it becomes continuous.

We present two different ways of redefinition. The first one is to consider the map f as a set-valued map, *i.e.* to set

$$f\left(\alpha^{-1}\right) = \{(\alpha, 1), (\alpha, 0)\} \text{ and } f\left(\beta^{-1}\right) = \{(\beta, 1), (\beta, 0)\}$$

With this definition map f becomes an upper semicontinuous set-valued map, it is also lower semicontinuous everywhere except at points α^{-1} and β^{-1}.

Another concept is closely related to the previous one, but instead of having two images of a troublesome point it splits the troublesome point into two (see also [8]).

We define the space extended $\widetilde{X_h}$ as consisting of all points which are not preimages of α^{-1} and β^{-1} plus for each point $\mathbf{x}$ such that $f^k(\mathbf{x}) = \alpha^{-1}$ or β^{-1} we distinguish $\mathbf{x}_-$ and $\mathbf{x}_+$. The first sort of points we call *two-sided* and the second is *one-sided* (or +- and −-points). Furthermore, we regard the following points as one-sided:

$$\begin{aligned} (\alpha, 1) &= \alpha_+ \qquad & (\beta, 0) &= \beta_- \\ (a, 0) &= a_+ \qquad & (b, 1) &= b_-. \end{aligned}$$

An ordering (for points with same state) is induced on $\widetilde{X_h}$ by the ordering of X_h with the addition $\mathbf{x}_- < \mathbf{x}_+$.

An extended mapping $\widetilde{f}$ is defined by

- if $\mathbf{x}$ is two-sided then $\widetilde{f}(\mathbf{x}) = f(\mathbf{x})$
- if $\mathbf{x}$ is one-sided then
  - $\widetilde{f}(\alpha_+^{-1}) = \alpha_+$, $\widetilde{f}(\alpha_-^{-1}) = (\alpha, 0)_{(-)}$ and the same for β^{-1}.
  - $\widetilde{f}(\alpha_+) = a_+$, $\widetilde{f}(\beta_-) = b_-$.
  - $\widetilde{f}(a_+) = f(a)_{(+)}$, $\widetilde{f}(b_-) = f(b)_{(-)}$.
  - $\widetilde{f}(\mathbf{x}_-) = f(\mathbf{x})_-$, $\widetilde{f}(\mathbf{x}_+) = f(\mathbf{x})_+$ for the other one-sided points.

A sign in parentheses is used only when the corresponding point is split.

The next step is to fix a metric on the extended space $\widetilde{X_h}$. The metric of X gives

$$\rho(\mathbf{x}_-, \mathbf{x}_+) = 0 \quad \text{therefore} \quad \mathbf{x}_- = \mathbf{x}_+,$$

which does not suit us. The new definition of the metric is closely related to the notion of the *kneading sequence* of a point. If $\mathrm{St}(\mathbf{x}) = \mathrm{St}(\mathbf{y})$ we put

$$\widetilde{\rho}(\mathbf{x}, \mathbf{y}) = \rho(\mathrm{Obs}(\mathbf{x}), \mathrm{Obs}(\mathbf{y})) + \sum_{i=1}^{\infty} 2^{-i} |\mathrm{St}\big(f^i(\mathbf{x})\big) - \mathrm{St}\big(f^i(\mathbf{y})\big)|, \qquad (4.1)$$

where $\rho(\cdot,\cdot)$ is the distance in $\mathbb{R}^1$. For $\mathbf{x}$ and $\mathbf{y}$ such, that $\mathrm{St}(\mathbf{x}) \neq \mathrm{St}(\mathbf{y})$ we set

$$\widetilde{\rho}(\mathbf{x},\mathbf{y}) = P + 1, \tag{4.2}$$

where

$$P = \sup_{\mathrm{St}(u)=\mathrm{St}(v)} \rho(u,v)$$

is the maximal distance between points of the same state.

Now to check triangle inequality in the case $\mathrm{St}(\mathbf{x}) = \mathrm{St}(\mathbf{y}) = \mathrm{St}(\mathbf{z})$ we observe, that

$$\begin{aligned} \rho(\mathrm{St}(\mathbf{x}), \mathrm{St}(\mathbf{y})) &\leq \rho(\mathrm{St}(\mathbf{x}), \mathrm{St}(\mathbf{z})) + \rho(\mathrm{St}(\mathbf{z}), \mathrm{St}(\mathbf{y})) \\ |\mathrm{St}\big(f^i(\mathbf{x})\big) - \mathrm{St}\big(f^i(\mathbf{y})\big)| &\leq |\mathrm{St}\big(f^i(\mathbf{x})\big) - \mathrm{St}\big(f^i(\mathbf{z})\big)| \\ &\quad + |\mathrm{St}\big(f^i(\mathbf{z})\big) - \mathrm{St}\big(f^i(\mathbf{y})\big)| \text{ for any } i, \end{aligned}$$

since ρ is a metric and $\mathrm{St}(f^i(\cdot))$ are real numbers. Summing these inequalities according to Eq. (4.1) we obtain the result for $\widetilde{\rho}$.

When states of points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are different we note, that the distance between any two points with the same state is not greater than $P+1$ and Eq. (4.2) ensures that the triangle inequality holds.

Lemma 7 *In the metric $\widetilde{\rho}$ the space $\widetilde{X_h}$ is compact and complete.*

Proof. Any sequence in X_h corresponds to a sequence in $\widetilde{X_h}$, which may be "larger" in the sense that some points of the first sequence correspond to two points of the second. We employ this correspondence to obtain our result.

First we prove the auxiliary statement: if a sequence $\{\mathbf{x}_n\} \in X_h$ converges to a point $\mathbf{x}$ in the space X_h then there is a subsequence $\{\mathbf{x}'_n\} \in \widetilde{X_h}$ which converges either to $\mathbf{x}$ or to one of $\mathbf{x}_\pm = \{\mathbf{x}_-, \mathbf{x}_+\}$.

There are two cases to consider, $\mathbf{x} \in X_h$ corresponds to one point $\mathbf{x} \in \widetilde{X_h}$ and $\mathbf{x}$ corresponds to two points, $\mathbf{x}_-$ and $\mathbf{x}_+$. In the latter case we choose a monotone subsequence $\{\mathbf{x}'_n\}$ which converges in X_h to $\mathbf{x}$ from one side. Let it be convergent from the left, without loss of generality. Assume, that this subsequence does not converge to $\mathbf{x}_-$ in $\widetilde{X_h}$. Then there must be a number i, such that $\mathrm{St}(f^i(\mathbf{x}_{n'})) \neq \mathrm{St}(f^i(\mathbf{x}_-))$ for any $n' > N$. Otherwise, since

$\mathrm{St}(f^i(\mathbf{x}_N)) = \mathrm{St}(f^i(\mathbf{x}_-))$ implies $\mathrm{St}(f^i(\mathbf{x}_{n'})) = \mathrm{St}(f^i(\mathbf{x}_-))$ for any $n' > N$ (for explanation see Chapter 6) we obtain

$$\sum_{i=1}^{\infty} 2^{-i} |\mathrm{St}\big(f^i(\mathbf{x}_{n'})\big) - \mathrm{St}\big(f^i(\mathbf{x}_-)\big)| \to 0,$$

which is a contradiction.

Thus we have $\mathrm{St}(f^i(\mathbf{x}_{n'})) \neq \mathrm{St}(f^i(\mathbf{x}_-))$ and it means that there is a i-preimage of a discontinuity point, $\mathbf{y}_\pm$, such that $\mathbf{x}_{n'} < \mathbf{y}_\pm < \mathbf{x}_-$. The points $\mathbf{y}_\pm$ correspond to a point $\mathbf{y} \in X_h$ and $\mathbf{x}_{n'} < \mathbf{y} < \mathbf{x}$, which contradicts our assumption that $\mathbf{x}_{n'} \to \mathbf{x}$.

It is interesting to note, that the sequence $\{\mathbf{x}_n\}$ may contain subsequences which converge to $\mathbf{x}_-$ and subsequences which converge to $\mathbf{x}_+$. This is not the case when $\mathbf{x}$ is not a preimage of a discontinuity point. However, the same argument as above can be used to prove the convergence $\mathbf{x}_{n'} \to \mathbf{x}$ and thus we omit it.

Now to prove the compactness of the space $\widetilde{X_h}$ we take an arbitrary sequence $\{\mathbf{x}_n\} \in \widetilde{X_h}$ and consider the corresponding sequence in the space X_h. The space X_h is compact and we find a subsequence which is convergent in X_h. Then, using the auxiliary statement, we choose a subsubsequence convergent in $\widetilde{X_h}$.

We make use of the same approach to prove that the space $\widetilde{X_h}$ is complete. From a Cauchy sequence, which converges in X_h, we choose a subsequence convergent in $\widetilde{X_h}$. But if a Cauchy sequence has a convergent subsequence it converges itself and $\widetilde{X_h}$ is complete. Q.E.D.

Each extension of couple (X_h, f) has its advantages and we make use of both of them. First (set-valued) approach is used in the rest of the current chapter and in Chapter 5 and the second approach is very convenient in the development of kneading theory, Chapter 6.

4.2 Maps with hysteresis and discontinuous maps

There are many different ways to establish correspondences between the dynamics of a map with hysteresis and the dynamics of a classical discontinuous interval map. We present two constructions of this type. However, we found it harder to study these classical maps than the original one.

4.2.1 Map with "mirrors"

A point $\mathbf{x}$ with $\mathrm{St}(\mathbf{x}) = 0$ is switched to state 1 by the map f if and only if $\mathbf{x} \in \Big(\beta^{-1}, (\beta, 0)\Big]$. Similarly, if $\mathrm{St}(\mathbf{x}) = 1$ and $x \in \Big[(\alpha, 1), \alpha^{-1}\Big)$, the state of the next iterate is $\mathrm{St}(f(x)) = 0$.

The idea of the map with mirrors is to put two functions f_0 and f_1 sufficiently far aside such that their domains of definitions do not intersect and to place two additional linear functions of the form $\mathbf{x} + c$, "mirrors", in order to transfer points from f_0 to f_1 and back. Thus, the interval $\Big(\beta^{-1}, (\beta, 0)\Big]$ is mapped onto the first "mirror" and then to the corresponding interval in the domain of the function f_1. This construction adds one step to the trajectory each time it switches from one state to the other, but trajectories of the original problem are in one-to-one correspondence with trajectories of the modernized map.

An example of a map with hysteresis and of a corresponding map with mirrors is presented on Fig. B.4.

4.2.2 First return map

Assume that a map with hysteresis has no fixed points. Then the dynamics is such that if we take a sufficiently large interval in the domain of definition of a function f_i, $i = 1, 2$, trajectories of points from the interval will eventually return to it. Examples of such intervals are $\Big(\beta^{-1}, (\beta, 0)\Big]$ and $\Big[(\beta, 1), (b, 1)\Big]$. The first choice has an advantage that no points are in the interval after the first iteration.

We define the map $g\colon J_\beta \to J_\beta$, $J_\beta = \Big(\beta^{-1}, (\beta, 0)\Big]$ by putting

$$g(\mathbf{x}) = f^{k(\mathbf{x})}(\mathbf{x}),$$

where $k(\mathbf{x}) = \min_{i>0} \{f^i(\mathbf{x}) \in J_\beta\}$.

An example of a first return map is given on Fig. B.5. As seen on the picture it has a regular structure with some discontinuity points. The structure is very similar to one of a NDI map [1]:

Definition 18 NDI *(N discontinuities, increasing) maps of the interval are those $h\colon [a,b] \to [a,b]$ satisfying:*

There exist $a < c_1 < c_2 < \cdots < c_N < b$ such that

1. *h is continuous and strictly increasing on (a, c_1), (c_1, c_2), ..., (c_N, b).*
2. *$\lim_{\mathbf{x} \to c_i -} h(\mathbf{x}) = b$ and $\lim_{\mathbf{x} \to c_i +} h(\mathbf{x}) = a$ for all $i = 1, \ldots, N$.*

Actually, the first return map of a map with hysteresis is a composition of two NDI maps:

Lemma 8 *An interval map $g\colon J \to J$, $J = (\mathrm{Obs}(\beta^{-1}), \beta]$, is the first return map of a map with hysteresis without fixed points if and only if there are NDI maps $h_1, h_2\colon J \to J$ that $g(\mathbf{x}) = h_2(h_1(\mathbf{x}))$.*

We consider *first visit* maps (defined in analogy to first return map) $\widetilde{h}_1$ from interval J to $J' = (\mathrm{Obs}(\beta^{-1}), \beta]$ and $\widetilde{h}_2\colon J' \to J$. It is easy to see that the maps are (after applying an homeomorphism from J' to J) NDI maps and, conversely, given two NDI maps there is a choice of a map with hysteresis, such that these maps are first visit maps.

4.3 Topologically expansive maps and conjugate maps

Here we introduce definitions specific to maps with hysteresis.

Definition 19 *A map with hysteresis f is said to be* topologically expansive *if for any points $\mathbf{x}$ and $\mathbf{y}$, which are not preimages of the discontinuity points, there is an iteration n such that*

$$\mathrm{St}(f^n(\mathbf{x})) \neq \mathrm{St}(f^n(\mathbf{y}))\,.$$

The following lemma gives the relation of this definition to the alternative one [1].

Lemma 9 *The following statements are equivalent:*

1. *Preimages of the points α^{-1} and β^{-1} are everywhere dense in X_h.*

2. *f is topologically expansive.*

3. *There exists $\epsilon > 0$ such that for any points $\mathbf{x}$ and $\mathbf{y}$, which are not preimages of the discontinuity points*

$$\rho\left(f^i(\mathbf{x}), f^i(\mathbf{y})\right) > \epsilon$$

for some i.

Proof. 1. $\Rightarrow$ 2. Let $\mathrm{St}(\mathbf{x}) = \mathrm{St}(\mathbf{y})$. Let k be the minimal number such that there is a k-preimage of a discontinuity point in the interval $(\mathbf{x}, \mathbf{y})$. Then $\mathrm{St}(f^i(\mathbf{x})) = \mathrm{St}(f^i(\mathbf{y}))$, $i = 1, \dots k$ and $\mathrm{St}\big(f^{k+1}(\mathbf{x})\big) \neq \mathrm{St}\big(f^{k+1}(\mathbf{y})\big)$.

2. $\Rightarrow$ 3. By the definition of the metric on X_h, $\mathrm{St}(f^i(\mathbf{x})) \neq \mathrm{St}(f^i(\mathbf{y}))$ implies that $\rho\left\{f^i(\mathbf{x}), f^i(\mathbf{y})\right\} > P$, where P is a constant.

3. $\Rightarrow$ 1. Here we use an argument similar to the one in [1]. Let A_1 and A_2 be the sets of preimages of the discontinuity points α^{-1} and β^{-1} respectively. We are going to prove that closure $\overline{A} = X$, where $A = A_1 \bigcup A_2$.

Assume the contrary, $B = X \setminus \overline{A}$ is nonempty. The set B is open by the definition, therefore it is a countable collection of intervals. Now we take an arbitrary interval $B_0 \subset B$ from the collection. The set B is invariant therefore B_0 is mapped by f to another interval, which we denote by B_1: $f\left(B_0\right) \subset B_1$. Proceeding by induction we get the sequence $\{B_i\}_{i=0}^{\infty}$, $f\left(B_i\right) \subset B_{i+1}$.

There are two possibilities to consider: either the sequence is periodic or the intervals B_i are all different. In the first case, f maps some interval into itself, which is incompatible with condition 3. In the second case, lengths of intervals will eventually become less than any ϵ which is also a contradiction. Q.E.D.

Definition 20 *Two maps with hysteresis f and g defined on spaces X_h and X'_h are said to be* topologically conjugate *if there is a state-preserving homeomorphism ϕ: $X_h \to X'_h$ such that*

$$\mathrm{Obs}(\phi\left(f(\mathbf{x})\right)) = \mathrm{Obs}(g(\phi(\mathbf{x}))) \ \textit{and} \ \mathrm{St}(f(\mathbf{x})) = \mathrm{St}(g(\phi(\mathbf{x})))\,. \qquad (4.3)$$

4.4 Continuity of the graph of L

Although we were able to prove upper semicontinuity of the graph of the global attractor L vs a parameter λ, a general map with hysteresis does not have other types of continuity (lower semicontinuity and measure-continuity).

Example 1 For the map shown on Fig. B.6 the set L is the whole interval $[a, \beta] \times \{0\}$ and three intervals on the branch 1. However, any increase of the parameter c (with the parameter d fixed) will result in disappearance of the interval $\left[\left(f_0^{-1}(d), 0\right), \left(f_0^{-1}(e), 0\right)\right]$ from the branch 0 after some iterations. Thus, the choice $\lambda = c$ causes both lower and measure discontinuity in $L(\lambda)$.

However, in some simple cases we can prove continuity of the graph. First we prove an auxiliary lemma.

Lemma 10 *The boundary of the global attractor is* $\partial(L) \subset \overline{\mathrm{Img}(\{\alpha^{-1}, \beta^{-1}\})}$.

Proof. For the boundary of the global attractor one has

$$\partial(L) \subset \overline{\bigcup_{i=0}^{\infty} \partial\left(f^i(X_h)\right)}$$

and, therefore, it is sufficient to prove that the boundary $\partial\left(f^i(X_h)\right)$ belongs to the set

$$\bigcup_{k=0}^{i} \left(f^k\left(\alpha^{-1}\right) \cup f^k\left(\beta^{-1}\right)\right)$$

for any i.

We prove it by induction. The boundary of $f^0(X_h)$ consists of the points α, β, $(a, 0) \in f(\alpha)$ and $(b, 1) \in f(\beta)$. Assume that the statement is proven for $f^i(X_h)$.

The closed set $f^{i+1}(X_h)$ is a finite collection of closed intervals. Let $\mathbf{x} \in \partial\left(f^{i+1}(X_h)\right)$. Then $\mathbf{x}$ has a preimage $\mathbf{y}$. If $\mathbf{y} \in \partial\left(f^i(X_h)\right)$ we are done by induction. In the other case $\mathbf{y}$ is a point of discontinuity of the function f, $\mathbf{y} = \alpha^{-1}$ or $\mathbf{y} = \beta^{-1}$. Indeed, assume the contrary: f is continuous at $\mathbf{y}$ and, therefore, monotone. Then there exists an open neighbourhood U, $\mathbf{y} \in U \subset f^i(X_h)$ such that f is continuous on U. Therefore, $f(U)$ is an open set and $\mathbf{x} \in f(U) \subset f^{i+1}(X_h)$. Thus we get $\mathbf{x} \notin \partial\left(f^{i+1}(X_h)\right)$, which is a contradiction.

Q.E.D.

Theorem 3 *Let* $\{f_\lambda\}_{\lambda\in\Lambda}$ *be a family of (set-valued) maps with hysteresis weakly upper continuous at* λ_0. *Let at the point* λ_0 *the set* L *be equal to* $f^n(X_h)$ *for some* n *and*

$$\begin{array}{rl} \alpha^{-1}\notin f^i\left(\alpha^{-1}\right), \quad \beta^{-1}\notin f^i\left(\beta^{-1}\right) & \quad 1\le i\le n+1 \\ f^i\left(\alpha^{-1}\right)\cap f^j\left(\beta^{-1}\right)=\emptyset & \quad 0\le i,j\le n+1. \end{array}$$

Then graph of L *is lower semicontinuous and measure-continuous at* λ_0.

Proof. To prove the theorem we develop a slightly new approach to the set L. For each iteration i we consider the set

$$B_i=\left\{\alpha^{-1},\beta^{-1},f\left(\alpha^{-1}\right),f\left(\beta^{-1}\right),\dots,f^{i+1}\left(\alpha^{-1}\right),f^{i+1}\left(\beta^{-1}\right)\right\}$$

of the possible boundaries of the set $f^i(X_h)$ (points α^{-1} and β^{-1} are not the possible boundaries, but we include them also). The set B_i depends on λ, which we indicate by writing $B_i(\lambda)$ sometimes. Conditions of the theorem imply that for each $0\le i\le n+1$ the set $B_i(\lambda_0)$ consists of exactly $4i+6$ points. Note, that $f\left(\alpha^{-1}\right)$ and $f\left(\beta^{-1}\right)$ are sets of two points each.

Now we divide intervals $[a,\beta]\times\{0\}$ and $[\alpha,b]\times\{1\}$ into subintervals with boundaries in B_i. For example, for $i=0$ subintervals are

$$\left[(a,0),\beta^{-1}\right],\ \left[\beta^{-1},(\beta,0)\right]\text{ on 0-branch}$$

and

$$\left[(\alpha,1),\alpha^{-1}\right],\ \left[\alpha^{-1},(b,1)\right]\text{ on 1-branch.}$$

Every time we obtain exactly $4i+4$ subintervals. We denote the set of subintervals by S_i. It is easy to see that the interior of any subinterval from S_i may either be a subset of $f^i(X_h)$ or be disjoint with it (otherwise there are boundary points in the interior which is in contradiction to the definition of S_i). We say that a subinterval $J\in S_i$ is full if $S_i\subset f^i(X_h)$. Otherwise we say that it is empty. For example, for $i=0$ there are no empty intervals.

Although we use here the first (set-valued) concept of a continuous map with hysteresis it is helpful to split points α^{-1} and β^{-1}. In other words, we put, for example

$$f\Big(\left[(\alpha,1),\alpha^{-1}\right]\Big)=[a,\alpha]\times\{0\}\ \text{ and }\ f\Big(\left[\alpha^{-1},(b,1)\right]\Big)=[\alpha,f_1(b)]\times\{1\}.$$

However, this addition is made to simplify the proof and has no effect on the dynamics of the map (it is implied of the fact that α^{-1} and β^{-1} are not in the set of possible boundaries of $f^i(X_h)$ for $0 \le i \le n+1$).

With this modernization, the properties of the set S_i are the following:

- For every $J \in S_i$ the function $f|_J$ is a monotone continuous function.
- Let $J_1 \in S_i$ and $J_2 \in S_{i+1}$ be such that $f(J_1) \cap J_2 \ne \emptyset$. Then $J_2 \subset f(J_1)$.

There is a certain partial ordering of the sets $B_i(\lambda)$, $i \le n+1$ and it is easy to see, that for small $\epsilon > 0$ the ordering for $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ is the same as for λ_0. The set B_{n+1} consists of finite number of points; we can choose δ such that δ-neighbourhoods of these points do not intersect. Then for each point $\mathbf{x}_{\lambda_0} \in B_{n+1}$ we find corresponding ϵ_x such that $\mathbf{x}_\lambda$, belongs to a δ-neighbourhood of $\mathbf{x}_{\lambda_0}$, provided, that $\lambda \in (\lambda_0 - \epsilon_x, \lambda_0 + \epsilon_x)$. The needed ϵ is minimum of ϵ_x over all $\mathbf{x} \in B_{n+1}$.

If the ordering is preserved then set S_i is preserved too. There is a natural one-to-one correspondence between $S_i(\lambda)$ and $S_i(\lambda_0)$. We argue that an interval from $S_i(\lambda)$ is full if and only if the corresponding interval from $S_i(\lambda_0)$ is full.

We prove it by induction. For $i = 0$ the statement is true. Assume it is true for $i-1$. If an interval $J_{\lambda_0} \in S_i(\lambda_0)$ is full then there is a full interval $J'_{\lambda_0} \in S_{i-1}(\lambda_0)$ such that $f\left(J'_{\lambda_0}\right) \supset J_{\lambda_0}$. Then the corresponding interval $J'_\lambda \in S_{i-1}(\lambda)$ is also full (by induction assumption) and preserved ordering of $B_i(\lambda)$ implies that interval corresponding to J_{λ_0}, J_λ, is $J_\lambda \subset f_\lambda(J'_\lambda)$. The same argument proves that if an interval J_λ is full, J_{λ_0} is full also.

Finally we obtain that the equality $f^{n+1}_{\lambda_0}(X_h) = f^n_{\lambda_0}(X_h)$ implies that $f^{n+1}_\lambda(X_h) = f^n_\lambda(X_h)$ and, therefore, $f^n_\lambda(X_h) = L(\lambda)$ with the intervals of $L(\lambda)$ in one-to-one correspondence to the intervals of $L(\lambda_0)$. We already know, that the Hausdorff distance between $B_n(\lambda)$ and $B_n(\lambda_0)$ can be made arbitrarily small and, therefore,

$$\rho\left(L(\lambda_0), L(\lambda)\right) \to 0 \quad \text{as } \lambda \to \lambda_0.$$

This observation implies lower semicontinuity and, since the number of intervals in $L(\lambda)$ is constant, measure-continuity. Q.E.D.

Chapter 5

Piecewise linear maps with hysteresis

5.1 Basic properties of the the PLMH

The piecewise linear map with hysteresis is a map with

$$f_0(x) = \gamma_0 x \quad \text{and} \quad f_1(x) = \gamma_1 x$$

and the threshold points are

$$\alpha = \frac{a}{\gamma_1} \quad \text{and} \quad \beta = \frac{b}{\gamma_0}$$

Lemma 11 *A piecewise linear map with hysteresis has periodic points if and only if $\gamma_0^k \gamma_1^l = 1$ for some integer k and l. If there are any periodic points then each point is eventually periodic.*

Proof. It easy to see that existence of periodic points implies that $\gamma_0^k \gamma_1^l = 1$. To prove the converse we consider all irreducible numbers of the form $x\gamma_0^i \gamma_1^j$, where (x, s) is a point from X_h. We call a number irreducible if and only if there are no i' and j' such that

$$i' < i, \quad j' < j \text{ and } \gamma_0^{i'} \gamma_1^{j'} = \gamma_0^i \gamma_1^j.$$

In other words, either i must be less than k or j less than l (otherwise take $i' = i - k$ and $j' = j - l$). This condition and condition $0 < a < x\gamma_0^i \gamma_1^j < b$ clearly imply that there is only a finite number of possibilities for i and j.

For any n we have $\mathrm{Obs}(f^n(x,s)) = x\gamma_0^i\gamma_1^j$ and, since there is only a finite number of possibilities, $f^n(x,s) = f^{n+k}(x,s)$ for some n and k. This observation finishes the proof.

Note that the period k and the transition n are uniformly bounded. Another way to formulate this lemma is to say, that a PLMH is periodic if and only if $\ln\gamma_1/\ln\gamma_0$ is rational. Observe that in this case $x \in L$ if and only if x is periodic (and not just eventually periodic).

Lemma 12 *If a piecewise linear map is obtained from another map by*

- *Multiplying the numbers a, b, α and β by a positive coefficient k.*
- *Raising the numbers a, b, α, β, γ_0 and γ_1 to a positive power p*

then these maps are topologically conjugate.

To prove this lemma we use homeomorphism $\phi(x,s) = (kx,s)$ in the first case and $\phi(x,s) = (x^p,s)$ in the second case.

Another useful property of PLMH is existence of the *non-increasing measure.*

Definition 21 *The measure μ is said to be* non-increasing *under a map f if for any open set U*

$$\mu(f(U)) \le \mu(U).$$

Lemma 13 *The measure $d\mu = d\ln x$ is non-increasing under a PLMH. Moreover, if $\mathrm{St}(\mathbf{x}) = \mathrm{St}(\mathbf{y})$ for any $\mathbf{x},\mathbf{y} \in U$ then $\mu(f(U)) = \mu(U)$*

We remind that for any $A \subset X_h$

$$A = (A_0 \times \{0\}) \cup (A_1 \times \{1\})$$

and A is measurable if A_0 and A_1 are measurable,

$$\mu(A) = \mu(A_0) + \mu(A_1) = \int I_{A_0} d\ln x + \int I_{A_1} d\ln x,$$

where I_{A_0} and I_{A_1} are indicator functions.

Now we consider a family of piecewise linear maps with hysteresis which are obtained by varying one of the parameters α, β, γ_0 or γ_1. This family is weakly continuous at every point and, therefore, the graph of $L(\lambda)$ as a function of the parameter is upper semicontinuous at every point. We cannot say the same about lower semicontinuity. However we observe that in the example of a non-continuous graph (see Chapter 4) a crucial role is played by a trajectory which connects two discontinuity points.

Conjecture 1 *The graph $L(\lambda)$ is lower semicontinuous if*

$$\alpha^{-1} \notin \mathrm{Img}\big(\beta^{-1}\big) \qquad \beta^{-1} \notin \mathrm{Img}\big(\alpha^{-1}\big).$$

We will prove the conjecture in the irrational case after learning some properties of the discontinuity points α^{-1} and β^{-1}.

There exists a possibility to classify the sets L basing on the number of discontinuities of the first return maps used in Lemma 8. The possible pairs of numbers are 0 and 1, 1 and 1, 1 and greater than 1. The simplest case is 0 and 1 or, in other words

$$\alpha\gamma_1^{-i} \notin [\beta, b] \text{ for any } i \quad \text{ or } \quad \beta\gamma_0^{-i} \notin [a, \alpha] \text{ for any } i.$$

Then the first return map to the interval $[\beta, b] \times \{0\}$ (in the first case) is just a circle homeomorphism [9] and the set L has the very simple structure,

$$L = \bigcup_{i=0}^{N} f^i([\beta, b], 0)$$

with some finite N. The example of such a map with the set L and the first return map is shown on Fig. B.7.

5.2 Principle of equivalent distance

Our subsequent analysis will be based mostly on the following Principle:

Theorem 4 (Principle of equivalent distance). *Let an interval $(\mathbf{x}, \mathbf{y}) \subset X_h$ contain no k-preimages of the discontinuity points, where $k = 1, \dots, K$. Then the set $f^k((\mathbf{x}, \mathbf{y}))$ is a connected open interval for $k = 1, \dots, K$ and*

$$C_1 r \le \rho\left(f^k\left(\mathbf{x}\right)_+, f^k\left(\mathbf{y}\right)_-\right) \le C_2 r,$$

where $r = \rho(\mathbf{x}, \mathbf{y})$, $k = 1, \dots, K$,

$$f^k(\mathbf{z}_0)_\pm = \lim_{\mathbf{z} \to \mathbf{z}_0 \pm} f^k(\mathbf{z})$$

and C_1, C_2 *are constants depending on* f *only.*

Proof. Since there are no preimages of the discontinuity points, f^k is continuous on $(\mathbf{x}, \mathbf{y})$. Moreover, f^k acts on observables as a linear function, $f^k(z) = \gamma_0^k \gamma_1^l z$, for some k and l. These observations settle the first part of the Principle.

The choice of the possible k and l is restricted since $f^k(z)$ does not leave the interval $[a, b]$. In other words, there is a number $z \in [a, b]$ such that $\gamma_0^k \gamma_1^l z \in [a, b]$. This implies that the inequalities

$$\gamma_0^k \gamma_1^l a \leq b \quad \text{and} \quad \gamma_0^k \gamma_1^l b \geq a$$

must be satisfied. We summarize the inequalities in

$$\frac{a}{b} \leq \gamma_0^k \gamma_1^l \leq \frac{b}{a}$$

and put $C_1 = a/b$, $C_2 = b/a$ to finish the proof.

5.3 Preimages of the discontinuity points

Theorem 5 *Let the slopes* γ_0 *and* γ_1 *be such that function* f *has no periodic points. Then the set of preimages of the discontinuity points* α^{-1} *and* β^{-1} *is everywhere dense in* X_h.

Proof. We assume the contrary and repeat the third part of the proof of Lemma 9 to get the sequence $\{B_i\}$.

Interval B_i is new for each i, i.e. $B_i \neq B_j$ when $i \neq j$. Otherwise, there exist i and j such, that $f^j(B_i) \subset B_i$. B_i does not contain preimages of threshold points, therefore f^j is continuous on B_i and has a fixed point. However f does not have periodic points and we get contradiction.

Now by the Principle of equivalent distance we have

$$\mu(B_j) \geq \mu\left(f^j(B_0)\right) \geq C_1 \mu(B_0)$$

for any j, where μ is (Lebesgue) measure. Therefore

$$\infty > \mu(X_h) \geq \sum_{i=0}^{\infty} \mu\left(B_i\right) \geq C_1 \sum_{i=0}^{\infty} \mu\left(B_0\right) = \infty$$

and we get a contradiction. Q.E.D.

5.4 Omega-limit sets of the discontinuity points

In this Section we study images of discontinuity points α^{-1} and β^{-1}.

Definition 22 *A finite set F is said to be an* ϵ-net *of a set A if*

$$A \subset \bigcup_{x \in F} B_\epsilon(x).$$

If A is compact and G is dense in A than one can choose an ϵ-net F, $F \subset G$.

The following lemmas are proven under assumption that $\ln \gamma_1 / \ln \gamma_0$ is irrational and the condition

$$\beta^{-1} \notin \mathrm{Img}\big(\alpha^{-1}\big), \qquad \alpha^{-1} \notin \mathrm{Img}\big(\beta^{-1}\big). \tag{5.1}$$

is satisfied.

Lemma 14 *Let the set of preimages of the point α^{-1} be everywhere dense. Then $L = \omega\left(\alpha^{-1}\right)$.*

Proof. The conditions of the lemma imply that for any point $\mathbf{x}$ the set $f^k(\mathbf{x})$ consists of two points at most. Indeed, map f is single-valued everywhere, except the points α^{-1} and β^{-1}. If a point $\mathbf{x}$ is a preimage of α^{-1} then the set $f^k(\mathbf{x})$ will consist of two values after some iterations. But further division is impossible, because $f^k(\mathbf{x})$ cannot be equal to α^{-1} again (f has no periodic points) and cannot be equal to β^{-1} due to condition (5.1).

The structure of the map f suggests that for any $\mathbf{x}_0$ two possible values of $f^k(\mathbf{x}_0)$ are

$$\lim_{\mathbf{x} \to \mathbf{x}_0 -} f^k(\mathbf{x}) \quad \text{and} \quad \lim_{\mathbf{x} \to \mathbf{x}_0 +} f^k(\mathbf{x}).$$

Now every point $\mathbf{y} \in L$ has a k-preimage $\mathbf{y}_k$ such that

$$\lim_{\mathbf{x} \to \mathbf{y}_k -} f^k(\mathbf{x}) = \mathbf{y} \quad \text{or} \quad \lim_{\mathbf{x} \to \mathbf{y}_k +} f^k(\mathbf{x}) = \mathbf{y}.$$

It is easy to see that for any $\epsilon > 0$ there is N such that n-preimages of the point α^{-1}, $n = 1, \ldots, N$ form a $\epsilon/2$-net of the space X_h. Let $\mathbf{y}_k$ be a k-preimage of a point $\mathbf{y}$, $k > N$ and $\mathbf{y}$ is the limit of $f^k(\mathbf{x})$ as $\mathbf{x} \to \mathbf{y}_k$ from the left, without loss of generality. The open interval $(\mathbf{y}_k - \epsilon, \mathbf{y}_k)$ contains at least one of the n-preimages of the point α^{-1}, $n = 1, \ldots, k$. Let α^{-j} be the nearest of these preimages. Then the interval $(\alpha^{-j}, \mathbf{y}_k)$ satisfies the conditions of Principle of equivalent distance and applying f^k we obtain

$$\lim_{\mathbf{x} \to \alpha^{-j}+} f^k(\mathbf{x}) = \mathbf{z} \in \mathrm{Img}\left(\alpha^{-1}\right), \quad \lim_{\mathbf{x} \to \mathbf{y}_k -} f^k(\mathbf{x}) = \mathbf{y}, \quad \text{and} \quad \rho(\mathbf{z}, \mathbf{y}) < C_2 \epsilon.$$

Since ϵ was arbitrary and C_2 is fixed we can find an image of α^{-1} in any neighbourhood of $\mathbf{y}$. Therefore, $\mathbf{y} \in \omega\left(\alpha^{-1}\right)$.

The converse, $\omega\left(\alpha^{-1}\right) \subset L$, is always true. Q.E.D.

Lemma 15 *Let the sets X_α and X_β of limit points of preimages of α^{-1} and β^{-1} be non-empty. Then $L = \omega\left(\alpha^{-1}\right) = \omega\left(\beta^{-1}\right)$.*

Proof. First of all, Theorem 5 implies that

$$X_h = X_\alpha \cup X_\beta.$$

It is easy to see that there are points p_1 and p_2 such that for any ϵ there are preimages of α^{-1} in the intervals $(p_1 - \epsilon, p_1)$ and $(p_2, p_2 + \epsilon)$ and preimages of β^{-1} in the intervals $(p_1, p_1 + \epsilon)$ and $(p_2 - \epsilon, p_2)$. Next we find the intervals

$$\begin{aligned}
\left(\alpha^{-j_1}, \beta^{-k_1}\right) &\subset (p_1 - \epsilon, p_1 + \epsilon) \qquad j_1 \le k_1 \\
\left(\alpha^{-j_2}, \beta^{-k_2}\right) &\subset (p_1 - \epsilon, p_1 + \epsilon) \qquad j_2 \ge k_2 \\
\left(\beta^{-j_3}, \alpha^{-k_3}\right) &\subset (p_2 - \epsilon, p_2 + \epsilon) \qquad j_3 \le k_3 \\
\left(\beta^{-j_4}, \alpha^{-k_4}\right) &\subset (p_2 - \epsilon, p_2 + \epsilon) \qquad j_4 \ge k_4
\end{aligned}$$

to satisfy Principle of equivalent distance.

Applying the function f^{n-1}, $n = \max\{j,k\}$, to each interval we get that α^{-1} is a limit point of images of β^{-1} with limiting sequences approaching form both left and right. The same is true about β^{-1}. Thus we have

$$\overline{\mathrm{Img}(\alpha^{-1})} \subset \omega\left(\beta^{-1}\right),$$
$$\overline{\mathrm{Img}(\beta^{-1})} \subset \omega\left(\alpha^{-1}\right).$$

However, observe, that $\omega(\mathbf{x}) \subset \overline{\mathrm{Img}(\mathbf{x})}$ for any $\mathbf{x}$ and, therefore, $\omega(\beta^{-1}) = \omega(\alpha^{-1})$. Now we repeat the proof of Lemma 14 to conclude that any $\mathbf{y} \in L$ is contained either in $\omega(\beta^{-1})$ or in $\omega(\alpha^{-1})$, but since they coincide we obtain

$$L = \omega\left(\beta^{-1}\right) = \omega\left(\alpha^{-1}\right).$$

The lemma is proven.

In the following theorem condition (5.1) is not compulsory.

Theorem 6 *Let* $\ln\gamma_1/\ln\gamma_0$ *be irrational. The set* $\mathrm{Img}(\{\alpha^{-1},\beta^{-1}\})$ *is everywhere dense in* L*:*

$$L \subset \overline{\mathrm{Img}(\{\alpha^{-1},\beta^{-1}\})}.$$

Proof. Proofs of the enclosure $\mathbf{y} \in \omega(\alpha^{-1}) \cup (\beta^{-1})$, as given in Lemmas 14 and 15, are still valid for any $\mathbf{y} \in L$ which is not an image of a discontinuity point even if condition (5.1) is violated. Therefore,

$$L = \omega\left(\alpha^{-1}\right) \cup \omega\left(\beta^{-1}\right) \cup \mathrm{Img}\left(\left\{\alpha^{-1},\beta^{-1}\right\}\right) \subset \overline{\mathrm{Img}(\{\alpha^{-1},\beta^{-1}\})}.$$

Q.E.D.

5.5 Main theorems

Now we can summarize the consequences of the previous sections.

Theorem 7 *If* $\ln\gamma_1/\ln\gamma_0$ *is irrational and condition (5.1) is satisfied then* $L = \omega(\mathbf{x})$ *for any* $\mathbf{x}$.

Proof. Union of sets X_α and X_β, defined in the Lemma 15, is the whole space X_h, therefore, for any $\mathbf{x}$ (for $\mathbf{x}$ equal a, b, α or β consider $f^2(\mathbf{x})$ instead) we can find intervals

$$(\mathbf{y}_1,\mathbf{x}) \quad \text{and} \quad (\mathbf{x},\mathbf{y}_2)\,; \qquad \rho(\mathbf{y}_1,\mathbf{x}) < \epsilon \quad \rho(\mathbf{x},\mathbf{y}_2) < \epsilon$$

where $\mathbf{y}_1$ and $\mathbf{y}_2$ are some preimages of the discontinuity points. Applying Principle of equivalent distance to the intervals we obtain that (at least one of) the discontinuity points are (is) contained in $\omega(\mathbf{x})$. Lemmas 14 and 15 now imply that $L \subset \omega(\mathbf{x})$. Conversely $\omega(\mathbf{x}) \subset L$ is always true and the theorem is proven.

Theorem 8 *A piecewise linear map is topologically expansive if and only if* $\ln \gamma_1 / \ln \gamma_0$ *is irrational.*

Indeed, if $\ln \gamma_1 / \ln \gamma_0$ is irrational then preimages of points α^{-1} and β^{-1} are everywhere dense. Conversely, if preimages are everywhere dense then the set of preimages must be infinite. When $\ln \gamma_1 / \ln \gamma_0$ is rational all points are eventually periodic with uniformly bounded periods and transitions, see Lemma 11. Let p be the longest period and t be the longest transition. Then the set of preimages of point α and β is

$$\bigcup_{i=0}^{p+t} f^{-i}(\{\alpha, \beta\}),$$

which is clearly finite and nowhere dense.

Theorem 9 *Let f be a piecewise linear map with hysteresis. We impose condition (5.1) in the case when* $\ln \gamma_1 / \ln \gamma_0$ *is irrational. Then the global attractor L is equal to the non-wandering set Ω.*

Lemma 6 implies that $\Omega \subset L$. To prove inclusion $L \subset \Omega$ we consider two cases: $\ln \gamma_1 / \ln \gamma_0$ is rational and it is not. In the former case $\mathbf{x} \in L$ if and only if $\mathbf{x}$ is periodic, therefore $L \subset \Omega$.

If $\ln \gamma_1 / \ln \gamma_0$ is irrational we employ Theorem 7 to conclude that $L = \omega(\mathbf{x}) \subset \Omega$.

5.6 Continuity of the graph of $L(\lambda)$

Theorem 10 *Let $\alpha^{-1}, \beta^{-1} \in L(\lambda)$ for any λ from some open neighbourhood of λ_0. Then the graph $L(\lambda)$ is lower semicontinuous at λ_0.*

Proof. For any sequence $\lambda_n \to \lambda_0$ and any point $\mathbf{x}_o \in L(\lambda_0)$ we have to find a sequence $\mathbf{x}_n \to \mathbf{x}_0$, $\mathbf{x}_n \in L(\lambda_n)$.

If $\mathbf{x}_0 \in L(\lambda_n)$ we are done. Assume, that $\mathbf{x}_0 \notin L(\lambda_n)$ for any n (without loss of generality). We define the sequence $\{k_n\}$ to satisfy

$$\mathbf{x}_0 \in f_{\lambda_n}^{k_n-1}(X_h) \quad \text{and} \quad \mathbf{x}_0 \notin f_{\lambda_n}^{k_n}(X_h).$$

Intervals J_n are the maximal intervals to satisfy

$$\mathbf{x}_0 \in J_n \subset f_{\lambda_n}^{k_n-1}(X_h) \setminus f_{\lambda_n}^{k_n}(X_h).$$

Boundaries of the intervals J_n are contained in the sets

$$\partial\left(f_{\lambda_n}^{k_n-1}(X_h)\right) \cup \partial\left(f_{\lambda_n}^{k_n}(X_h)\right)$$

and, therefore, $\partial(J_n) \subset \mathrm{Img}(\alpha^{-1}\beta^{-1}) \subset L(\lambda_n)$ with the last inclusion implied by the condition of the theorem.

Now, if $\partial(J_n) \to \mathbf{x}_0$ we are done (we found a sequence $\mathbf{x}_n \in L(\lambda_n)$, $\mathbf{x}_n \to \mathbf{x}_0$). Assume that this is not true: there is a subsequence $\{n'\}$ ($= \{n\}$ without loss of generality) such that $\mu(J_n) > m > 0$. Then there are two cases to consider:

- $\{k_n\}$ is unbounded. Then for each interval J_n there is a sequence of $k_n - 1$ sets
$$J_n^{-i} \subset f_{\lambda_n}^{k_n-i}(X_h) \setminus f_{\lambda_n}^{k_n-i+1}(X_h)$$
such that $f\left(J_n^{-i}\right) = J_n^{-i+1}$. It is clear, that these sets are disjoint and if μ is a non-increasing measure, we have $\mu\left(J_n^{-i}\right) > \mu(J_n) > m$, which is contrary to the assumption that $\{k_n\}$ is unbounded.

- $\{k_n\}$ is bounded. Without loss of generality we assume that $k_n = k$ for any n. Then we consider a k-preimage of point $\mathbf{x}_0$ under f_{λ_0}, point $\mathbf{x}^{-k}$. Weak lower continuity and Lemma 2 imply that $f_{\lambda_n}^{k}\left(\mathbf{x}^{-k}\right) \to \mathbf{x}_0$ and we get a contradiction.

Therefore, the case $\partial J_n \not\to \mathbf{x}_0$ is impossible. Q.E.D.

Theorem 11 *Let a family of PLMH with the parameter λ be weakly continuous at a point λ_0, $\ln\gamma_1/\ln\gamma_0$ be irrational and*

$$\beta^{-1} \notin \mathrm{Img}\left(\alpha^{-1}\right), \qquad \alpha^{-1} \notin \mathrm{Img}\left(\beta^{-1}\right).$$

Then the graph $L(\lambda)$ is lower semicontinuous at the point λ_0.

Proof. First we prove an auxiliary statement: if $\alpha^{-1} \notin L(\lambda_0)$ then there is a neighbourhood of λ_0 such that for any λ from the neighbourhood, $\alpha^{-1} \notin L(\lambda)$.

Let k be such that

$$\alpha^{-1} \in f_{\lambda_0}^{k-1}(X_h) \text{ and } \alpha^{-1} \notin f_{\lambda_0}^{k}(X_h).$$

Then there is σ such that

$$\forall \lambda \left(|\lambda - \lambda_0| < \sigma\right) \Big(\alpha^{-1} \notin f_{\lambda}^{k}(X_h)\Big).$$

Indeed, assuming the contrary we obtain that

$$\exists \lambda_n \to \lambda_0 \ \exists \mathbf{z}_n \to \mathbf{z}\Big(\alpha^{-1} \in f_{\lambda_n}^{k}(\mathbf{z}_n)\Big)$$

and, by weak upper continuity, $\alpha^{-1} \in f_{\lambda_0}^{k}(\mathbf{z})$. This is a contradiction.

As a corollary we obtain that at least one of the points α^{-1} and β^{-1} is contained in the set L. Indeed, if $\ln\gamma_1/\ln\gamma_0$ is irrational, Theorem 5 implies that one of these points has an infinite number of preimages and, therefore, belongs to L. In the rational case we assume the contrary: both points are not in the set L. Then we choose γ_0 as a parameter and employ our auxiliary statement to deduce that $\alpha^{-1}, \beta^{-1} \notin L(\lambda)$ in some neighbourhood of λ. But irrational maps are dense in this neighbourhood and we get a contradiction.

Now to prove the theorem we consider two cases.

$\beta^{-1} \in L(\lambda_0)$ and $\alpha^{-1} \notin L(\lambda_0)$. Then $L(\lambda_0) = \omega(\beta^{-1})$ (Lemma 14). For any point $\mathbf{x} \in L(\lambda_0)$ there is an image of β^{-1} which is close to $\mathbf{x}$:

$$\forall \epsilon > 0 \ \exists \widetilde{\mathbf{x}} \in f_{\lambda_0}^{n}\lambda_0\left(\beta^{-1}\right) \Big((\widetilde{\mathbf{x}} - \mathbf{x}) < \epsilon/2\Big).$$

Lemma 2 implies that there is σ such that

$$\forall \lambda \left(|\lambda - \lambda_0| < \sigma\right) \ \exists \mathbf{x}_\lambda \in f_{\lambda}^{n}\left(\beta^{-1}\right) \Big(|\mathbf{x}_\lambda - \widetilde{\mathbf{x}}| < \epsilon/2\Big)$$

and, therefore, $|\mathbf{x}_\lambda - \mathbf{x}| < \epsilon$. Provided that $\mathbf{x}_\lambda \in L(\lambda)$ it is proof of the lower semicontinuity of the graph.

To prove that $\mathbf{x}_\lambda \in L(\lambda)$ it is sufficient to prove that $\beta^{-1} \in L(\lambda)$. But our auxiliary statement implies that $\alpha^{-1} \notin L(\lambda)$ for λ in some neighbourhood of λ_0 and, using the corollary, we conclude that $\beta^{-1} \in L(\lambda)$.

If both α^{-1} and β^{-1} are contained in $L(\lambda_0)$ then $L(\lambda_0) = \omega(\beta^{-1}) = \omega(\alpha^{-1})$. Therefore, we can perform the same analysis for both α^{-1} and β^{-1} to get

$$\forall\lambda\,(|\lambda-\lambda_0|<\sigma_1)\ \exists\mathbf{x}_\lambda \in f_\lambda^n\left(\alpha^{-1}\right)\Big(|\mathbf{x}_\lambda - \mathbf{x}| < \epsilon\Big)$$
$$\forall\lambda\,(|\lambda-\lambda_0|<\sigma_2)\ \exists\mathbf{y}_\lambda \in f_\lambda^n\left(\beta^{-1}\right)\Big(|\mathbf{y}_\lambda - \mathbf{x}| < \epsilon\Big).$$

Now, since either α^{-1} or β^{-1} belong to $L(\lambda)$ we deduce that either $\mathbf{x}_\lambda$ or $\mathbf{y}_\lambda$ belong to $L(\lambda)$ too. Q.E.D.

5.7 The graph of $L(\beta)$

We consider the graph of L which is obtained by varying the threshold β, $\lambda = \beta$. We consider the case when

$$\gamma_0^k \gamma_1^l = 1$$

for some mutually prime k and l. Let γ be such that

$$\gamma_0 = \gamma^k, \qquad \gamma_1 = \gamma^{-k}.$$

For each value of λ the set $L(\lambda)$ consists of finite number of closed intervals. Furthermore, the number of intervals is uniformly bounded if λ belongs to some bounded interval.

Lemma 10 implies that for any λ

$$\partial(L(\lambda)) \subset \mathrm{Img}(\alpha) \cup \mathrm{Img}(\beta) \subset \left\{\gamma^i\alpha : -k \le i \le k'\right\} \cup \left\{\gamma^i\beta : l' \le i \le l\right\},$$

where

$$k' = \max_i \left\{\gamma^i\alpha \le b\right\} \qquad l' = \min_i \left\{\gamma^i\beta \ge a\right\}.$$

We are going to prove that boundaries of the graph of $L(\lambda)$ are also contained in this set.

Lemma 16

$$\partial(\mathrm{Graph}(L(\lambda))) \subset \left\{\left(\lambda, \gamma^i\alpha\right) : -k \le i \le \infty\right\} \cup \left\{\left(\lambda, \gamma^i\lambda\right) : -\infty \le i \le l\right\}.$$

Proof. In this and the subsequent proofs we will use ideas from the proof of Lemma 3.

First we denote the set of possible boundaries by B,

$$B = \left\{ \left(\lambda, \gamma^i \alpha\right) : -k \le i \le \infty \right\} \cup \left\{ \left(\lambda, \gamma^i \lambda\right) : -\infty \le i \le l \right\}.$$

Set B consists of countable (and finite on any finite interval) number of straight lines and is shown on Fig. B.8 for $\gamma = 8/7$, $k = 3$ and $l = 4$.

To prove the lemma we assume the contrary. There is a point $(\lambda_0, \mathbf{x}) \in \partial(\mathrm{Graph}(L(\lambda))$ which is not in the set B.

The properties of this point are

- $\mathbf{x} \in L(\lambda_0)$, because the graph is closed.
- $f^n_{\lambda_0}(\mathbf{x}) = \mathbf{x}$ for some n, because all motion on $L(\lambda_0)$ is periodic.
- $f^i_{\lambda_0}(\mathbf{x}) \notin B$ for any i, because B is invariant under f_{λ_0}.

Let $\epsilon > 0$ be such that

$$\left(\lambda, f^i_{\lambda_0}(\mathbf{x})\right) \notin B \text{ for any } 0 \le i < n \text{ and } \lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon). \tag{5.2}$$

Then one has $f^i_{\lambda_0}(\mathbf{x}) = f^i_{\lambda}(\mathbf{x})$. Indeed, by induction, let $f^{i-1}_{\lambda_0}(\mathbf{x}) = f^{i-1}_{\lambda}(\mathbf{x})$. Eq. (5.2) implies that $f^{i-1}_{\lambda}(\mathbf{x})$ satisfies the same inequalities with respect to α and β $(= \lambda)$ as $f^{i-1}_{\lambda_0}(\mathbf{x})$ does and, therefore, the action of f is the same on both points. For $i = n$ this property yields $f^n_{\lambda}(\mathbf{x}) = f^n_{\lambda_0}(\mathbf{x}) = \mathbf{x}$, therefore, $\mathbf{x}$ is periodic under f_λ, is contained in $L(\lambda)$ and $f^i_\lambda(\mathbf{x}) \in L(\lambda)$ for any i.

Now let $\sigma < \epsilon$ be such that

$$(\mathbf{y}, \lambda) \cap B = \emptyset,$$

where $\mathbf{y} \in B_\sigma\left(f^i_{\lambda_0}(\mathbf{x})\right)$ for some $i < n$ and $\lambda \in B_\sigma(\lambda_0)$. Then for any such λ the set $\left\{(\lambda, \mathbf{y}) : \mathbf{y} \in B_\sigma\left(f^i_{\lambda_0}(\mathbf{x})\right)\right\}$ either belongs to $L(\lambda)$ or does not intersect with it. But we already know that $f^i_\lambda(\mathbf{x}) \in L(\lambda)$, therefore the whole set

$$\left\{(\lambda, \mathbf{y}) : \ \mathbf{y} \in B_\sigma\left(f^i_{\lambda_0}(\mathbf{x})\right), \ \lambda \in B_\sigma(\lambda_0)\right\} \subset \mathrm{Graph}(L(\lambda)$$

and point $(\lambda_0, \mathbf{x})$ is not boundary point of the graph. We get a contradiction.

Corollary 2 *The graph of $L(\lambda)$ is measure-continuous.*

For a given λ_0 the indicator functions $I_{L(\lambda)}$ converge to $I_{L(\lambda_0)}$ as $\lambda \to \lambda_0$ pointwise everywhere except possible boundary points (*i.e.* almost everywhere). Indeed, for any $\mathbf{y} \in L(\lambda_0) \setminus \partial(L(\lambda_0))$ the previous lemma implies that $\mathbf{y} \in L(\lambda)$ if λ is close to λ_0. The difference $I_{L(\lambda_0)}(\mathbf{y}) - I_{L(\lambda)}(\mathbf{y})$ is, therefore, 0. By the Dominated Convergence Theorem, $|I_{L(\lambda)} - I_{L(\lambda_0)}| \to 0$ in measure and this is equivalent to measure-continuity.

Lemma 17 *If*

$$f^i_{\lambda_0}\left(\alpha^{-1}\right) \not\ni \beta^{-1} \text{ and } f^i_{\lambda_0}\left(\beta^{-1}\right) \not\ni \alpha^{-1}$$

for any i then the graph of $L(\lambda)$ is lower semicontinuous at the point λ_0.

Proof. This lemma is an extension of Lemma 3 for the special case of periodic PLMH and varying threshold β. Indeed, the only violated condition of Lemma 3 is

$$\alpha^{-1} \notin f^i\left(\alpha^{-1}\right), \quad \beta^{-1} \notin f^i\left(\beta^{-1}\right) \qquad 1 \le i \le n+1, \tag{5.3}$$

because the dynamics of the map is periodic and the period of α^{-1} or β^{-1} might be less then n, where n is determined by the condition $f^n(X_h) = L$.

However, we made assumption (5.3) in order to ensure that the sets B_i (see proof of Lemma 3) are preserved under small changes of the parameter λ. Now the nature of the problem is such, that trajectories of α^{-1} and β^{-1} may change only if, for example, $f^i\left(\alpha^{-1}\right) = \beta^{-1}$ for some i. But this case is excluded by the condition of the lemma. Thus we can apply the proof of Lemma 3 to our case. Q.E.D.

Chapter 6

Kneading invariant of maps with hysteresis

6.1 Definition of kneading invariants

Throughout this chapter we will use the second concept of a continuous map with hysteresis and regard X_h and f as an extended space and map. The map f is assumed to be topologically expansive.

For a point $\mathbf{x} \in X_h$ we define the kneading sequence as a binary sequence

$$k(\mathbf{x}) = s_0 s_1 s_2 \cdots,$$

where $s_i = \mathrm{St}(f^i(\mathbf{x}))$.

We order kneading sequences lexicographically, i.e. $s_0 s_1 \cdots < r_0 r_1 \cdots$ if and only if there is $j \geq 0$ such that $s_i = r_i$ for $i < j$ and $s_j < r_j$. This ordering can be obtained also by writing a sequence as a number in base 2,

$$[k(\mathbf{x})] = \sum_{i=0}^{\infty} s_i 2^{-(i+1)}.$$

It is easy to see, that in this definition of the ordering kneading sequences are monotone in $\mathbf{x}$: $k(\mathbf{x}) \leq k(\mathbf{y})$ whenever $\mathbf{x} < \mathbf{y}$. The definition of topologically expansive map implies

Lemma 18 $k(\mathbf{x}) = k(\mathbf{y})$ *if and only if* $\mathbf{x} = \mathbf{y}$.

Corollary 3 *If the kneading sequence of a point* $\mathbf{x}$ *is periodic then* $\mathbf{x}$ *is also periodic.*

Now we define the shift operator σ:

$$\sigma(s_0 s_1 \cdots) = s_1 s_2 \cdots.$$

Its action clearly corresponds to the action of f on the original point $\mathbf{x}$,

$$\sigma(k(\mathbf{x})) = k(f(\mathbf{x})).$$

The most important kneading sequences for our analysis are

$$\begin{array}{rclcrcl} \bar{a} & = & k(a_+), & & \bar{b} & = & k(b_-), \\ \bar{\alpha} & = & k\left((\alpha,0)_{(-)}\right), & & \bar{\beta} & = & k\left((\beta,1)_{(+)}\right). \end{array}$$

Together they form the *kneading invariant* of the map f.

Next we define three types of condition, for the two-sided points, the $-$-points and the $+$-points. A kneading sequence $\bar{x} = k(\mathbf{x})$ satisfies a *middle* condition (C) if

$$\begin{array}{rcl} \bar{a} & < & \sigma^i(\bar{x}) < \bar{b}, \\ \sigma^i(\bar{x}) & < & \bar{\alpha} \quad \text{if } \sigma^{i-1}(\bar{x}) = 10\cdots, \\ \sigma^i(\bar{x}) & > & \bar{\beta} \quad \text{if } \sigma^{i-1}(\bar{x}) = 01\cdots, \end{array} \tag{C}$$

for $i > 1$. *Lower* (C$-$) and *upper* (C$+$) conditions are the similar conditions for the $-$-points and the $+$-points

$$\begin{array}{rcl} \bar{a} & < & \sigma^i(\bar{x}) \leq \bar{b}, \\ \sigma^i(\bar{x}) & \leq & \bar{\alpha} \quad \text{if } \sigma^{i-1}(\bar{x}) = 10\cdots, \\ \sigma^i(\bar{x}) & > & \bar{\beta} \quad \text{if } \sigma^{i-1}(\bar{x}) = 01\cdots, \end{array} \tag{C$-$}$$

$$\begin{array}{rcl} \bar{a} & \leq & \sigma^i(\bar{x}) < \bar{b}, \\ \sigma^i(\bar{x}) & < & \bar{\alpha} \quad \text{if } \sigma^{i-1}(\bar{x}) = 10\cdots, \\ \sigma^i(\bar{x}) & \geq & \bar{\beta} \quad \text{if } \sigma^{i-1}(\bar{x}) = 01\cdots, \end{array} \tag{C$+$}$$

The meaning of these conditions is simple: each successive image of a point under the mapping f (and, therefore, kneading sequence of the image) must lie between a and b — first inequality; when the state switches from 1 to 0, the point must be somewhere between a and α — second inequality — and when state switches to 1, the point must be between β and b — third inequality. Some inequalities are strict because a two-sided point can be mapped only to two-sided, a $-$-point cannot be mapped to $+$-point *etc.*

Lemma 19 *Every point* $\mathbf{x} \in X_h$ *satisfies corresponding condition. Conversely, for any sequence* $\bar{x}$*, satisfying one of the conditions (C), (C–) or (C+) there is a point* $\mathbf{x} \in X_h$ *such that* $k(\mathbf{x}) = \bar{x}$*.*

Proof. The first part of the Lemma is already proven. To prove the second part we assume that $\bar{x} = 1s_1s_2\ldots$ (without loss of generality) and consider the points

$$\mathbf{y}_1 = \sup_{k(\mathbf{y}) \leq \bar{x}} \mathbf{y} \quad \text{and} \quad \mathbf{y}_2 = \sup_{k(\mathbf{y}) \geq \bar{x}} \mathbf{y}.$$

Since $k(\alpha, 1) < \bar{x} < \bar{b}$ (the first inequality is implied by $\sigma(k(\alpha, 1)) = \bar{a} < \sigma(\bar{x})$) points $\mathbf{y}_1$ and $\mathbf{y}_2$ are well defined. It is easy to see that only two cases are possible: $\mathbf{y}_1 = \mathbf{y}_2 = \mathbf{y}$ and $\mathbf{y}_1 = \mathbf{y}_-$, $\mathbf{y}_2 = \mathbf{y}_+$ for some $\mathbf{y} \in X_h$.

Now we refer to the auxiliary statement, formulated in the proof of Lemma 7. It can be rephrased as follows

$$\lim_{z \to z_{(-)}-} k(z) = k(z_{(-)}), \qquad \lim_{z \to z_{(+)}+} k(z) = k(z_{(+)}),$$

where limits are understood in the topology, induced by the ordering. Thus, $k(\mathbf{y}_1) \leq \bar{x} \leq k(\mathbf{y}_2)$. We want to prove that either $k(\mathbf{y}_1) = \bar{x}$ or $k(\mathbf{y}_2) = \bar{x}$. Assume the contrary: inequalities are strict. Case $\mathbf{y}_1 = \mathbf{y}_2 = \mathbf{y}$ is therefore excluded. The only possibility is $k(\mathbf{y}_-) < \bar{x} < k(\mathbf{y}_+)$, $\mathbf{y}$ is a preimage of a discontinuity point. Let $f^i(\mathbf{y}) = \alpha^{-1}$. The states of $f^k(\mathbf{y}_-)$ and $f^k(\mathbf{y}_+)$ coincide for $k \leq i$. We apply σ^{i+1} to the inequality to obtain

$$k(f^{i+1}(\mathbf{y}_-)) < \sigma^{i+1}(\bar{x}) < k(f^{i+1}(\mathbf{y}_+))$$

and

$$\bar{\alpha} = k(\alpha, 0) < \sigma^{k+1}(\bar{x}) < k(\alpha, 1).$$

The first inequality is impossible if $\sigma^{k+1}(\bar{x}) = 0\ldots$ and the second one is impossible if $\sigma^{k+1}(\bar{x}) = 1\ldots$ (it implies $\sigma^{k+2}(\bar{x}) < \sigma(k(\alpha, 1)) = \bar{a}$). We get a contradiction.

Corollary 4 *Two topologically expansive maps with hysteresis have the same kneading invariant if and only if they are topologically conjugate.*

Proof. Definition 20 implies that kneading invariants of conjugate maps are equal.

To prove the converse we put: $h(\mathbf{x}) = \mathbf{x}'$ if and only if $k(\mathbf{x}) = k(\mathbf{x}')$, where $\mathbf{x} \in X_h$ and $\mathbf{x}' \in X_h'$. It is easy to see that h is continuous and continually invertible. Thus h is the homeomorphism needed in Definition 20. The corollary is proven.

Now we can state our main theorem (compare to [1]):

Theorem 12 *Let F be a topologically expansive map with hysteresis with kneading invariant $(\bar{a}, \bar{b}, \bar{\alpha}, \bar{\beta})$. Then $\bar{a}$ and $\bar{\beta}$ satisfy (C+), $\bar{b}$ and $\bar{\alpha}$ satisfy (C−).*

Conversely, for any kneading sequences $\left(\bar{a}, \bar{b}, \bar{\alpha}, \bar{\beta}\right)$ satisfying (C+) and (C−) respectively there exists a topologically expansive map f with hysteresis with the given kneading invariant and f is unique up to conjugacy.

6.2 Proof of Theorem 12

An observation made in Lemma 8 simplifies the proof in the case when f does not have fixed points (topologically expansive map with hysteresis can have only two fixed points: $f(a) = a$ and $f(b) = b$). In the absence of fixed points the theorem is just a corollary of the similar result for NDIE maps (N discontinuities, increasing expansive maps) [1].

However, when there are fixed points this approach is not easily applicable. Thus we prefer to give our own variant of the proof, suitable for any case. It is based on the ideas of the proof given in [1], but since conditions (C−) and (C+) are more strict than the corresponding conditions in [1] their implementation is slightly more difficult.

The first part of the Theorem and uniqueness in the second part are proven in the previous Lemma and its Corollary. To prove the rest of the second part we use the correspondence between kneading sequences and numbers written in base 2 to construct a mapping on a circle which is conjugate to the original mapping f.

The mapping on a circle is induced by the shift operator:

$$\sigma(x) = 2x \bmod 1.$$

We need to choose those points of the circle which correspond to possible kneading sequences. Therefore, these points must satisfy one of the conditions (C), (C−) or (C+), where σ is now a function on the circle and the inequalities are considered in the sense of real numbers. To choose these points we iterate the following algorithm:

The initial values are $W_1^1 = [\bar{a}, \bar{\alpha}]$ and $W_2^1 = [\bar{\beta}, \bar{b}]$.

1. $$\begin{aligned} G_1^{j+1} &= [\bar{a}, 1/2] \cap \textstyle\bigcup_{i=1}^{\infty} W_2^j/2^i \\ G_2^{j+1} &= [1/2, \bar{b}] \cap \textstyle\bigcup_{i=1}^{\infty} \left(W_1^j + 2^i - 1\right)/2^i \end{aligned}$$

2. $$\begin{aligned} W_1^{j+1} &= [\bar{a}, \bar{\alpha}] \cap G_1^{j+1} \\ W_2^{j+1} &= [\bar{\beta}, \bar{b}] \cap G_2^{j+1} \end{aligned}$$

Here kneading sequences are considered as numbers written in base 2. This algorithm is constructed to choose appropriate images of the intervals W_1^1 and W_2^1 under the 2-valued function σ^{-1}.

We consider the limit sets of the algorithm:

$$\begin{aligned} G_1 &= \lim_{j\to\infty} G_1^j, & G_2 &= \lim_{j\to\infty} G_2^j \\ W_1 &= \lim_{j\to\infty} W_1^j, & W_2 &= \lim_{j\to\infty} W_2^j, \end{aligned}$$

which satisfy the following properties:

1. $$\begin{aligned} G_1 &= [\bar{a}, 1/2] \cap \textstyle\bigcup_{i=1}^{\infty} W_2/2^i \\ G_2 &= [1/2, \bar{b}] \cap \textstyle\bigcup_{i=1}^{\infty} \left(W_1 + 2^i - 1\right)/2^i \end{aligned}$$

2. $$\begin{aligned} W_1 &= [\bar{a}, \bar{\alpha}] \cap G_1 \\ W_2 &= [\bar{\beta}, \bar{b}] \cap G_2 \end{aligned}$$

3. $$\begin{aligned} W_1 &\subset G_1 \\ W_2 &\subset G_2 \end{aligned}$$

Proposition 1 *Any point $c \in G_1 \cup G_2$ satisfies one of the conditions (C), (C−) or (C+).*

Indeed, let point c belong to set G_1. Then, by the property 1, $c \in 2^{-k_1}W_2$ for some k_1. Applying σ successively we get

$$\begin{aligned} \sigma^j(c) &\in 2^{-k_1+j}W_2 \subset [\bar{a}, 1/2] \quad \text{for } j < k_1 \\ \sigma^{k_1}(c) &\in W_2 \subset [\bar{\beta}, \bar{b}] \subset [1/2, \bar{b}]. \end{aligned}$$

Thus the conditions are clearly met for the first k_1 iterations. Furthermore, $\sigma^{k_1}(c) \in W_2 \subset G_2$, therefore $\sigma^{k_1}(c) \in 2^{-k_2}W_1 + \left(2^{k_2} - 1\right) 2^{-k_2}$ and we proceed by induction.

Proposition 2 *Any point $c \in [0,1] \setminus G_1 \cup G_2$ does not satisfy the conditions.*

Let the point c belong to $[0, 1/2]$, $c \in G_1^{j-1}$ and $c \notin G_1^j$. Then $c \in 2^{-k}W_2^{j-2}$ for some k and one has

$$\begin{aligned} \sigma^k(c) &\in W_2^{j-2} \subset G_2^{j-2} \\ \sigma^k(c) &\notin W_2^{j-1} \text{ therefore } \sigma^k(c) \notin G_2^{j-1}. \end{aligned}$$

We proceed by induction until the process ends in the situation

$$\sigma^n(c) \in W_\pm^0 \text{ and } \sigma^n(c) \notin W_\pm^1.$$

The point $\sigma^n(c)$ is thrown out after the first iteration and clearly does not satisfy the conditions. Therefore, c does not satisfy the conditions too. The proposition is proven.

Note, that the points $\bar{a}$, $\bar{b}$, $\bar{\alpha}$ and $\bar{\beta}$ are in the set $G_1 \cup G_2$, because they satisfy the conditions. Similarly, if c is a preimage of one of these points (under σ) and $c \in G_1^1 \cup G_2^1$ then $c \in G_1 \cup G_2$ also.

Proposition 3 *Set $[0,1] \setminus G_1 \cup G_2$ consists of disjoint open intervals without common endpoints.*

Assume that c is a common endpoint of two open intervals, therefore $c \in G_1 \cup G_2$ and c is isolated.

We say that a point x is the $+$-boundary of a set S if $x \in S$ and $(x - \epsilon, x) \cap [0,1] \setminus S \neq \emptyset$. Analogically, x is the $-$-boundary if $x \in S$ and $(x, x +$

$\epsilon) \cap [0,1] \setminus S \neq \emptyset$. An example is the point $\bar{a}$ which is $+$-boundary of the set G_1.

It is easy to see that all $-$-boundaries of the sets $G_1^k \cup G_2^k$ are preimages of $\bar{\alpha}$ or b under σ. The same is true about $+$-boundaries and the points $\bar{\beta}$ and a. Now we conclude that c cannot be isolated after a finite number of steps: otherwise it is preimage of ($\bar{\alpha}$ or b) and ($\bar{\beta}$ or a) and, therefore, c is a $+$-point and a $-$-point simultaneously which is not acceptable.

Thus $c \in J_k \subset G_1^k \cup G_2^k$ for any k, where J_k is an isolated closed interval and

$$\bigcap_{k=0}^{\infty} J_k = c. \tag{*}$$

However the remark we made after Proposition 2 implies that boundaries of J_k are contained in $G_1^k \cup G_2^k$. On the other hand, Eq. (*) implies that $\partial(J_k) \to c$, where $\partial(J_k)$ is the boundary of J_k. Therefore, c is not isolated. The proposition is proven.

Now to construct a map with hysteresis on an interval we use monotone bijections to map

$$\begin{aligned}
h_1 &: G_1 \cap [\bar{a}, \bar{\alpha}] \to [a, \alpha] \times \{0\} \\
h_2 &: G_2 \cap [\bar{\beta}, \bar{b}] \to [\beta, b] \times \{1\} \\
h_3 &: G_1 \cap [\bar{\alpha}, 1/2] \to [\alpha, \beta] \times \{0\} \\
h_4 &: G_2 \cap [1/2, \bar{\beta}] \to [\alpha, \beta] \times \{1\}
\end{aligned}$$

Remark 1 *The sets G_1 and G_2 contain entire intervals only in the degenerate case $\bar{a} = 0$, $\bar{b} = 1$ and $\bar{\alpha} = \bar{\beta} = 1/2$. Otherwise, images of this interval under σ will eventually cover the whole circle.*

To show that bijections are possible we propose a simple way to construct, for example, h_1. We represent set $[\bar{a}, \bar{\alpha}] \setminus G_1$ as a union of a countable number of open intervals $\bigcup_{i=1}^{\infty} U_i$. Then we identify the first interval U_1 with some point in $[a, \alpha]$, say, $(a+\alpha)/2$, interval U_2 with the point $(a + 3\alpha)/4$ or $(3a+\alpha)/4$, depending on the position of U_2 with respect to B_1 etc. Thus we establish one-to-one correspondences between intervals U_i and binary rationals of interval $[a, \alpha]$. Intervals U_i are dense in $[\bar{a}, \bar{\alpha}] \cap G_1$ (in the sense of Remark 1) and we extend the correspondence by continuity.

Note, that the boundaries of open intervals are mapped into one point x in the interval $[a, \alpha]$, but this corresponds to splitting x into x_- and x_+. Thus bijection is established between $G_1 \cap [\bar{a}, \bar{\alpha}]$ and the extended interval $[a, \alpha]$.

Finally we define the branch f_0 of a map with hysteresis by putting

$$\phi_0(x) = \begin{cases} \sigma\left(h_1^{-1}(x)\right) & \text{if } x \in [a, \alpha] \times \{0\} \\ \sigma\left(h_3^{-1}(x)\right) & \text{if } x \in [\alpha, \beta] \times \{0\} \end{cases}$$

$$f_0(x) = \begin{cases} h_1\left(\phi_0(x)\right) & \text{if } \phi_0(x) \in [\bar{a}, \bar{\alpha}] \\ h_3\left(\phi_0(x)\right) & \text{if } \phi_0(x) \in [\bar{\alpha}, 1/2] \\ h_2\left(\phi_0(x)\right) & \text{if } \phi_0(x) \in [\bar{\beta}, b] \end{cases}$$

and the function f_1 is defined analogously. Thus we constructed a map with hysteresis and it is an easy corollary of the procedure of the construction that kneading invariant of the map is the given $(\bar{a}, \bar{b}, \bar{\alpha}, \bar{\beta})$. This observation finishes the proof.

Chapter 7

Summary

In the work we studied a special case of multistate maps, interval maps of with hysteresis. We developed a theory for general maps with hysteresis as well as for a simple example, a piecewise linear map with hysteresis. The main object of our study was the global attractor L, or, in other words, the limit image of the space X under the map f.

The global attractor was shown to play a significant role in the dynamics of the map f. In the piecewise linear case (with some additional requirements) the set L turned to be the omega-limit set of any point and, therefore, nonwandering set of the map. We were able to prove continuity of the set L with respect to a parameter λ. In addition to upper semicontinuity in the general case, the set $L(\lambda)$, considered as a set-valued function of the parameter, is lower semicontinuous in a number of special cases. A conjecture formulated in Section 5.1 is a topic for future research. Other possible topics are: classification of types of L based on first return maps, formulation of sufficient condition of discontinuity and a study of the applicability of our technique to the case of general maps with hysteresis.

A part of the work was devoted to the study of combinatorial properties of maps with hysteresis. A natural extension of our results obtained is to define the renormalisation operator [2] for such maps. This will possibly simplify the classification of global attractor types.

Appendix A

C Programme

```
/* Three commline args: name of the output file, number
of preliminary iterations, number of valid iterations.
Start point is varying. Slope is also varying. */

#include<stdio.h>
#include<math.h>

int i;
long int st, it;
float x0, xs, a=1.35358, b, trsh1=1.0, trsh2=1.9;
FILE *outf;
char stat='0';

char Iter(void);

main(int argc, char *argv[])
{
  if (argc < 3) return 0;
  if ( (outf=fopen(argv[1], "w")) == NULL)
    puts("Hrenovo s failom! Error opening file!");

  st=atoi(argv[2]);
  it=atoi(argv[3]);
```

```
  for(b=0.46; b<0.7; b+=0.002){
    /* slope is varying here */

    for(xs=0.6; xs<1.2; xs+=0.05) {
      /* st is start point for iteration */

      stat='0'; x0 = xs;
      for(i=1; i<st; i++) stat=Iter();
      /* preliminary iterations */

      for(i=1; i<it; i++)
        {
          stat=Iter();
          /* valid iterations */

          if (stat == '1')
            fprintf(outf,"%f  %f\n",b, x0);
          /* Print observables when state is 0 */
        }
    }
  }
  fclose(outf);
}


/* iteration function */
char Iter(void)
{
  if ( (stat=='0') ) x0*=a; else x0*=b;

  if (x0 > trsh2) return '1';
  if (x0 < trsh1) return '0';
  /* return new state */
  return stat;
}
```

Appendix B

Figures

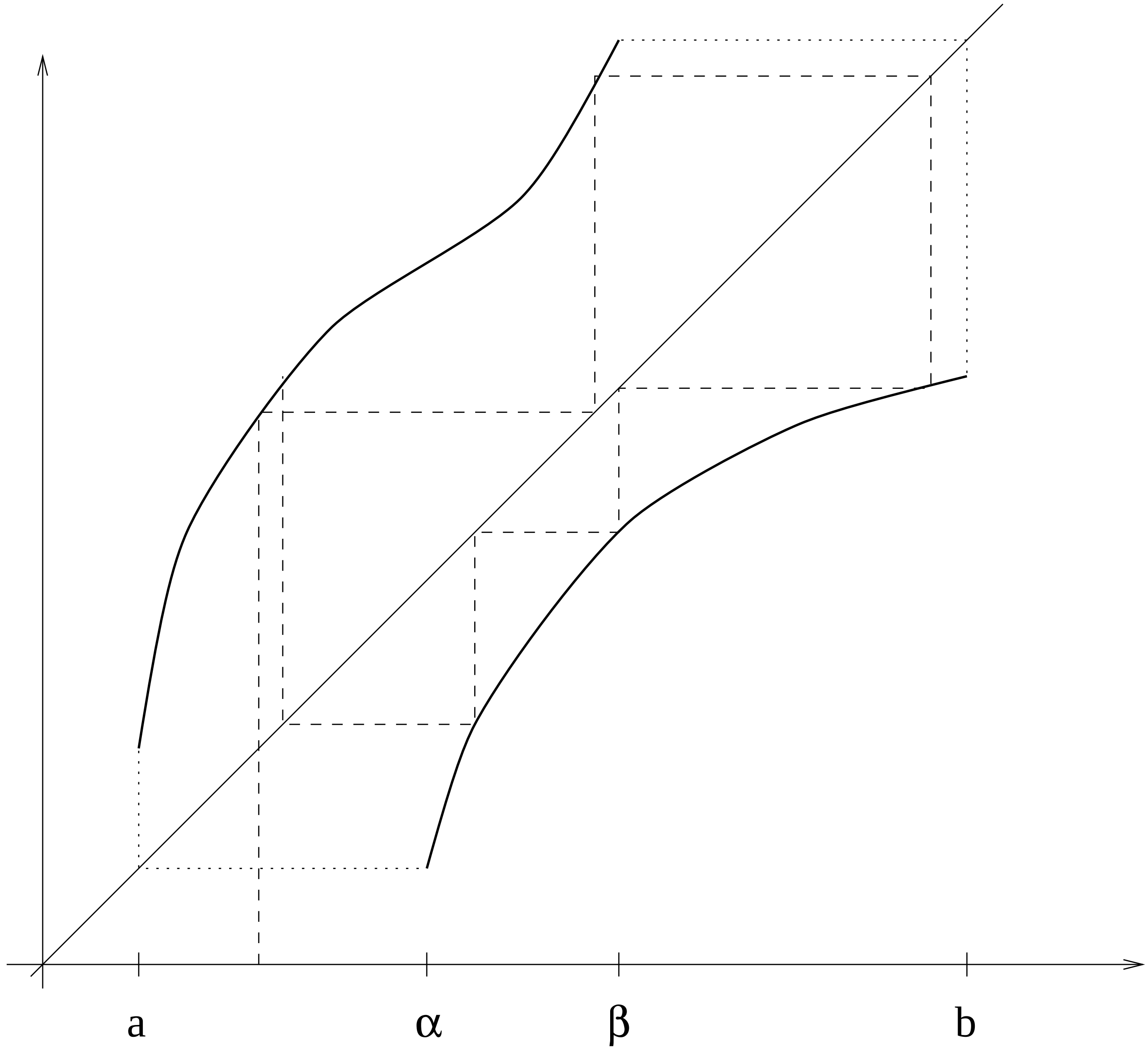

Figure B.1: An example of a map with hysteresis and a typical trajectory

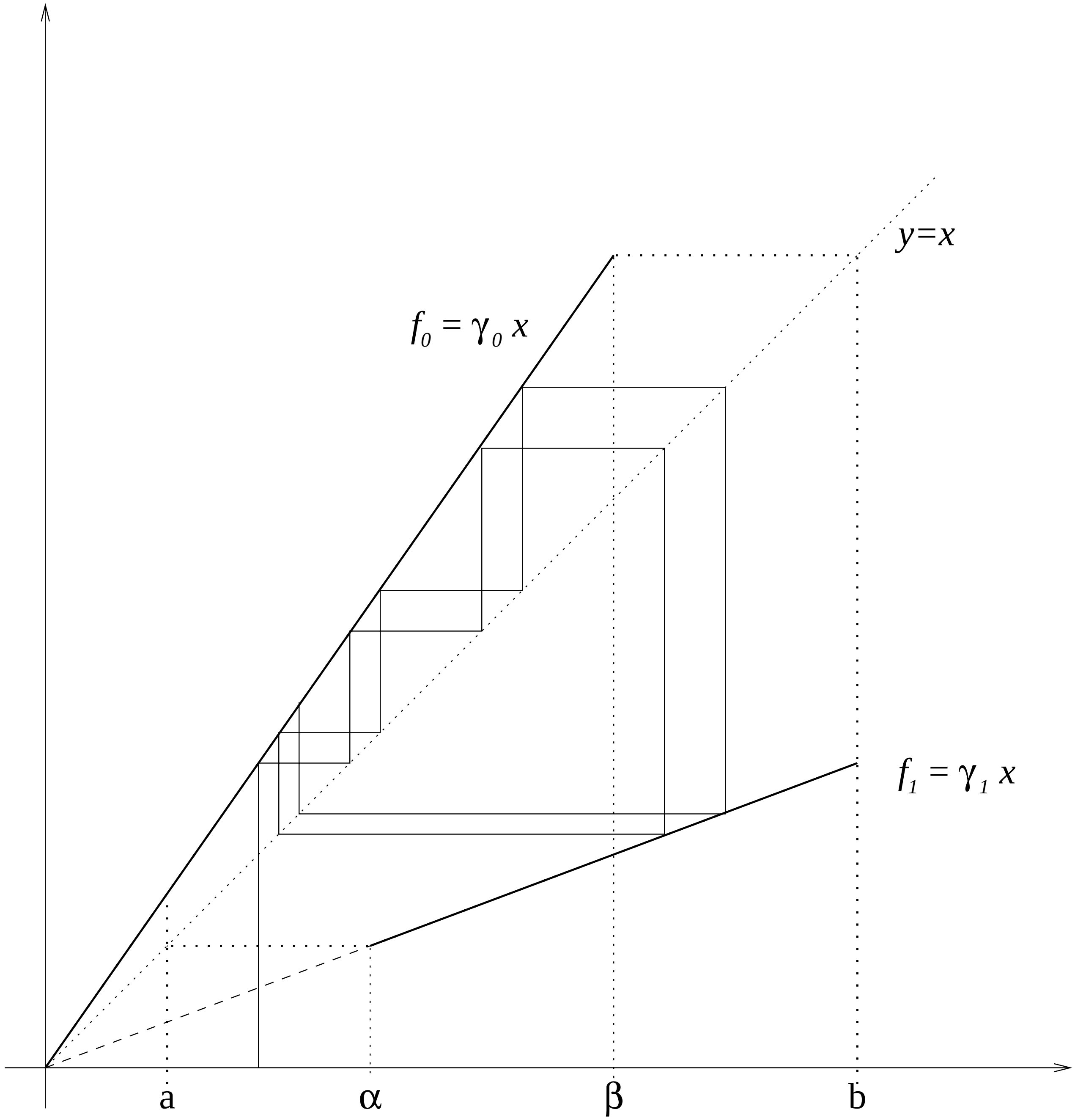

Figure B.2: An example of a piecewise linear map with hysteresis and a typical trajectory

Figure B.3: Graph of the global attractor of a piecewise linear map with hysteresis. The varying parameter is β.

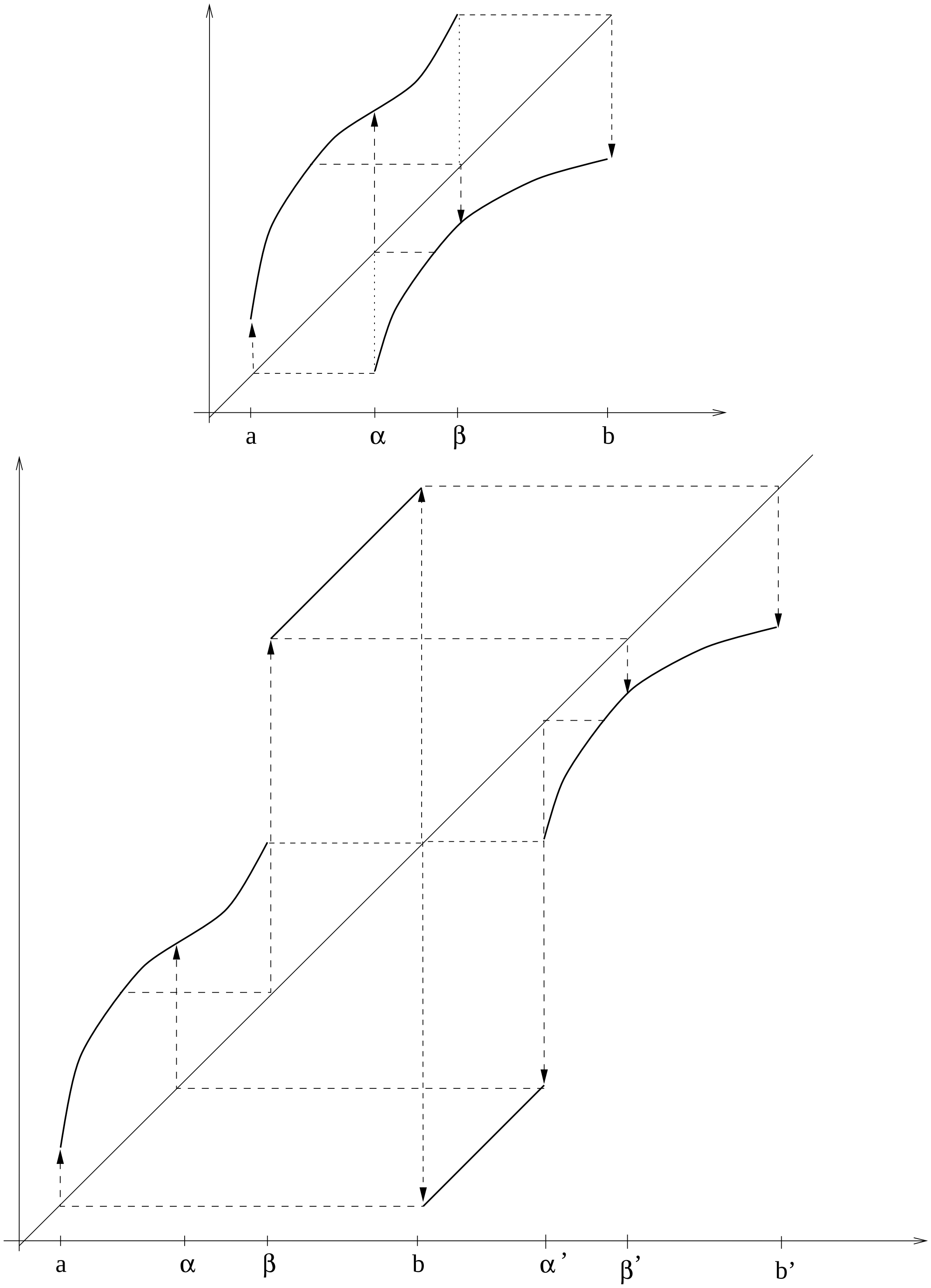

Figure B.4: A map with hysteresis and the corresponding map with "mirrors".

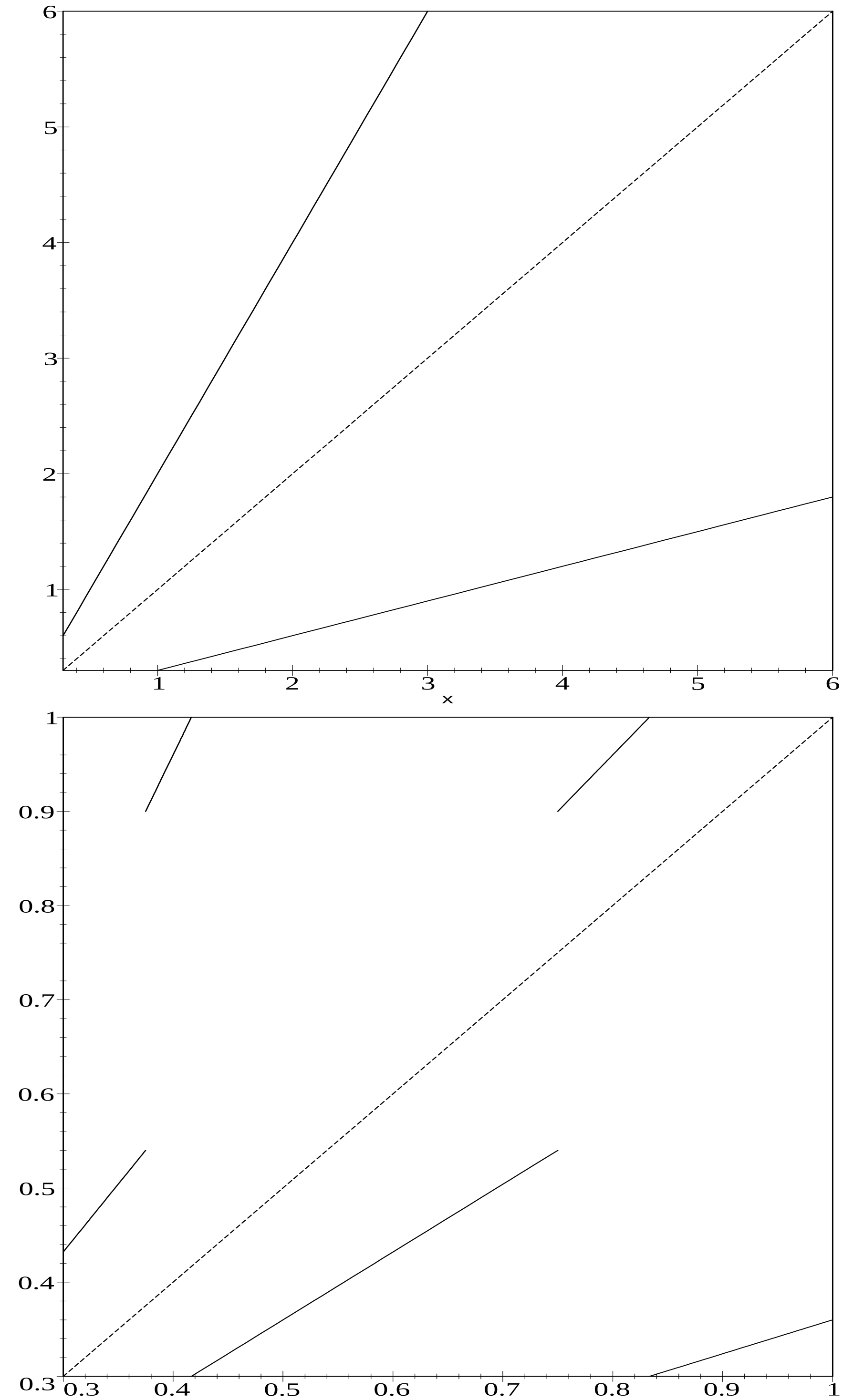

Figure B.5: A map with hysteresis, $\alpha = 1.0$, $\beta = 3.0$, $\gamma_0 = 2.0$, $\gamma_1 = 0.3$ and the first return map of interval $[0.3, 1.0]$.

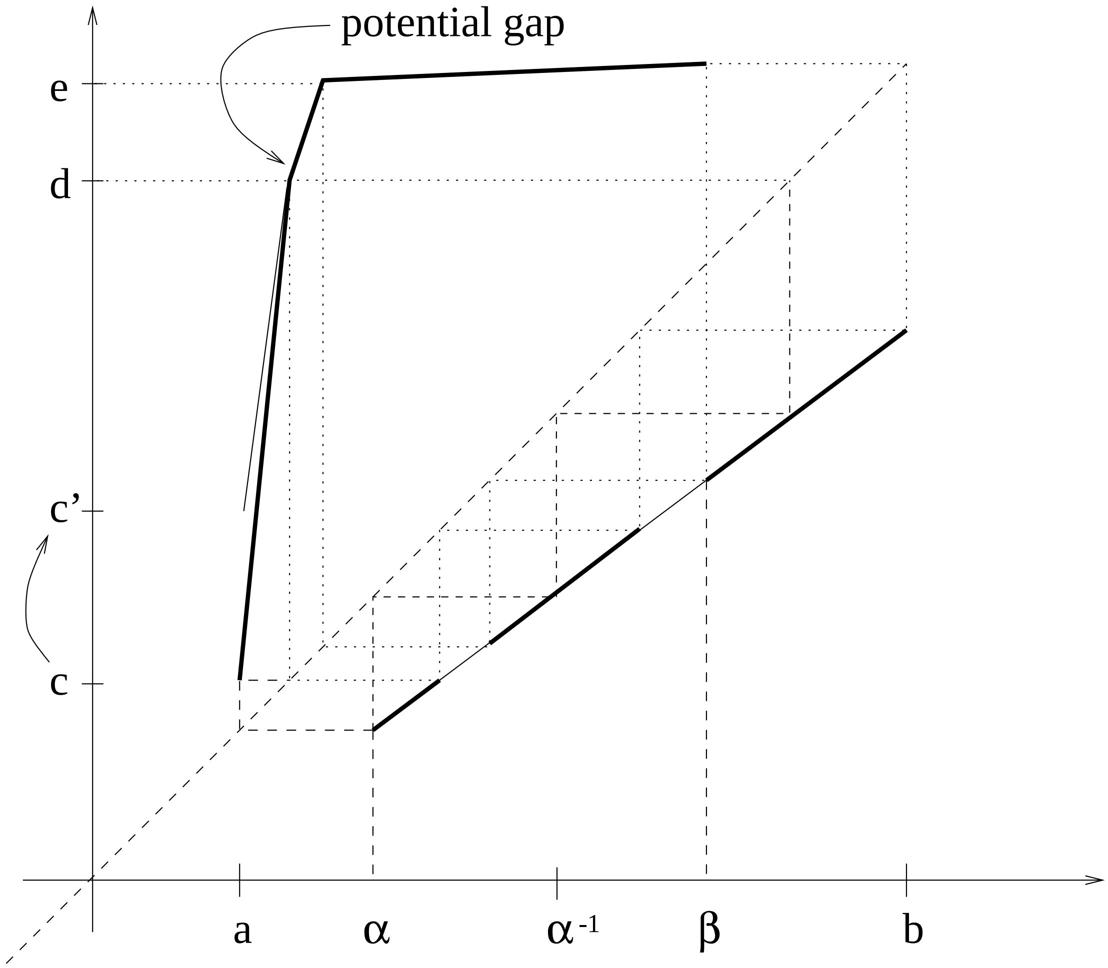

Figure B.6: An example of a family of maps which produce discontinuous graph. If we choose c as a parameter, with d fixed, the global attractor $L(c)$ is both measure and lower discontinuous.

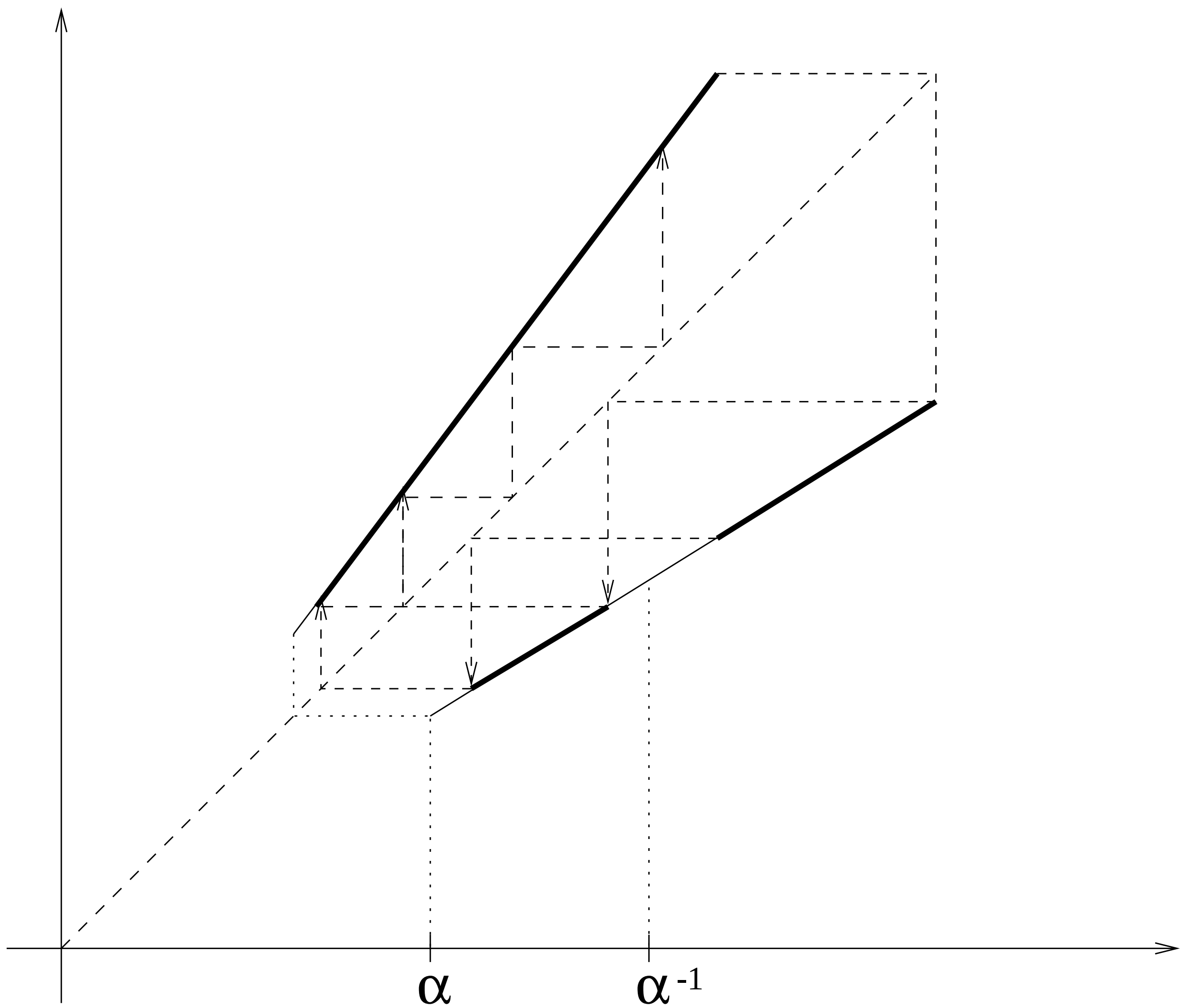

Figure B.7: The simplest example of a piecewise linear map with hysteresis. The first return map to the interval $[\beta, b]$ is a circle homeomorphism. The global attractor consists of one interval on the branch 0 and two intervals on the branch 1.

Figure B.8: Graph of the global attractor when $\gamma_0 = (8/7)^3$, $\gamma_1 = (7/8)^4$ and the varying parameter is β. On the second plot lines $y = (8/7)^k$ and $y = (8/7)^k x$ are added.

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