Number of distinct sites visited by Lévy flights injected into a d-dimensional lattice

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We study the average number of distinct sites $\langle S_{N_0}(t)\rangle$ visited by Lévy flights injected in the center of a lattice: N_0 new particles appear in the center of the lattice at each time step. Lévy flights are particles which have the probability $p(\ell)=A\ell^{-(1+\alpha)}, 0<\alpha<2$ of making an ℓ-length jump. We show analytically that the asymptotic form of $\langle S_{N_0}(t)\rangle$ is related to that of the case of constant initial number N of particles. We find that different ranges of α correspond to different limits, $t\to\infty$ and $N\to\infty$, in the behavior of the number of sites visited by *constant* number of particles. The results obtained analytically are in good agreement with Monte Carlo simulations. We also discuss possible results for $\alpha\geqslant 2$. [S1063-651X(98)03002-5]

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I. INTRODUCTION

Lévy flight distributions have been found recently in various areas of science [1–3]. Laboratory experiments in fluid flows reveal anomalous diffusion as a result of Lévy distribution [4,5], different species of animals searching for food have been found to follow the pattern of Lévy flights [6,7], and surface diffusion and turbulence are among the areas where scientists meet this distribution [8,9]. Mathematically it is expressed by a random walk model with sizes of steps taken from the distribution $p(\ell)=A\ell^{-(1+\alpha)}$ which is characterized by the high probability of large values of ℓ (compared to Gaussian distribution) and by the divergence of the second moment of single displacement (for $\alpha\leqslant 2$).

In this work we study a statistical property of Lévy flight, the average number of sites visited by random particles. This quantity is widely used in the Smoluchowski model for chemical reactions [10,11], in ecology, and in spreading phenomena [12–15]. Thus the problem of calculating this quantity is of great interest [16–27]. In our earlier work [16] we studied the number of distinct sites visited by N particles, each having Lévy probability $p(\ell)=A\ell^{-(1+\alpha)}$ to perform an ℓ-length jump. The particles are initially in the center of the lattice. We considered a constant number N of particles performing a t-step Lévy flight. Here we consider the case where the number of particles increases each time step with the constant rate N_0.

It has been found [8] that there exists anomalous diffusion at liquid surfaces, i.e., molecules execute excursions on the surface with displacements obeying Lévy distribution. The model where reactant A is injected at the rate N_0 (see also [28]) and diffuses over the surface of the substrate B producing the reaction $A(\text{diffusing})+B(\text{substrate})\to A(\text{diffusing})+C(\text{inert})$ corresponds to the problem of distinct sites visited by an increasing number of particles.

A similar problem was studied recently by Berezhkovskii and Weiss [27] for the case of Gaussian random walks in continuous time. Here we study the problem in discrete time for the case where a single jump has Lévy distribution or, to be more exact, a *stable law* distribution with infinite second moment.

Stable law distributions possess the characteristic property of retaining their form under summation. This means that distribution of random particles after time t has different scale but the same functional form. The highest term of the expansion of the *stable law* distribution with infinite second moment is $A\ell^{-(1+\alpha)}$, where $0<\alpha<2$ [29–31].

We study the problem of calculation of territory covered (or number of sites visited) using the approach presented by Larralde *et al.* [26] and further developed in [16]. Let $p_t(\mathbf{r})$ be the probability that a random walker is at site $\mathbf{r}$ at time step t, and $f_t(\mathbf{r})$ be the probability that a random walker visits site $\mathbf{r}$ at step t for the first time. The function

$$\Gamma_t(\mathbf{r})=1-\sum_{\tau=1}^{n} f_\tau(\mathbf{r}) \tag{1}$$

is the probability that a random walker has *not* visited $\mathbf{r}$ by the time t. Since N_0 particles appear each time step in the origin we can write down an expression for the expected number of distinct sites visited in the form

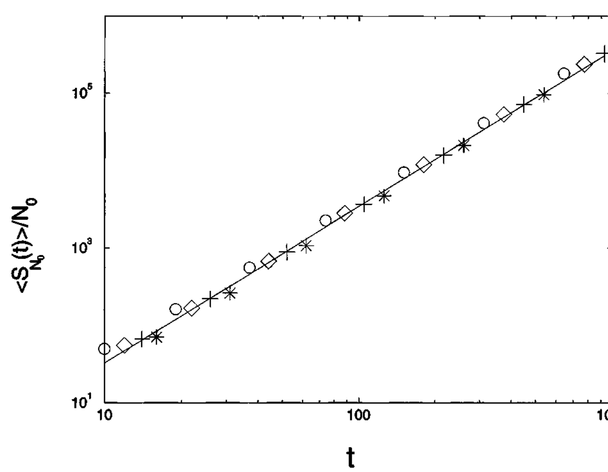

FIG. 1. Plot of the normalized function $\langle S_{N_0}(t)\rangle/N_0$ for $\alpha=0.4$. Different symbols correspond to values of $N_0=1(\bigcirc)$, 5 ($\diamond$), 10 (+), and 20 (*). The slope of the fitted line is 2.017 which is to be compared to the predicted value 2.0.

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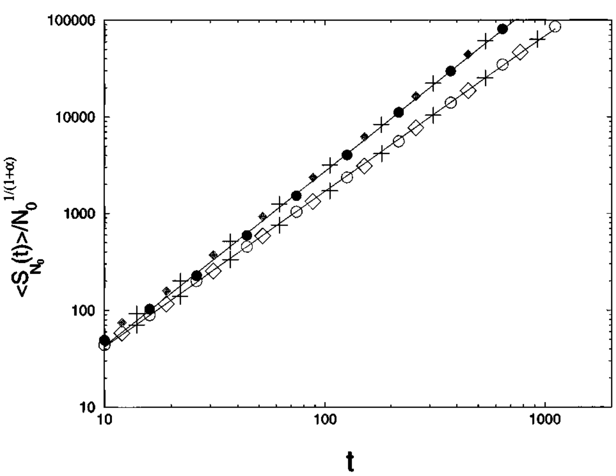

FIG. 2. Results of MC simulations for $\alpha=0.65$ (filled symbols) and $\alpha=0.8$ (empty symbols) show good agreement with the predicted behavior. The fitted slopes are 1.807 and 1.607, respectively, which are to be compared with the values $3/(1+\alpha)=1.818$ and 1.667. The plotted data correspond to $N_0=10$ (○), 50 (◇), and 100 (+).

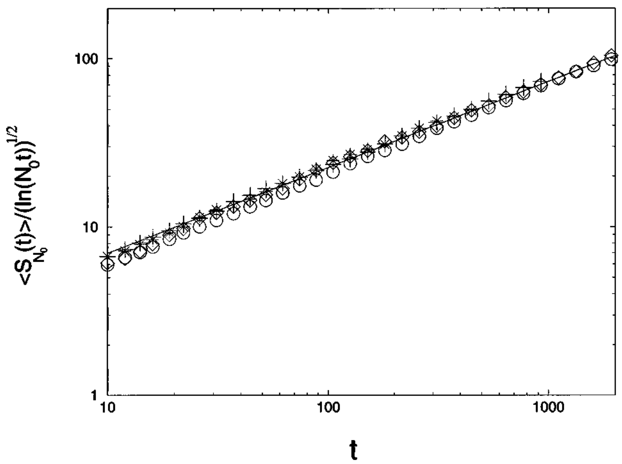

FIG. 3. Simulations for $\alpha=1.5$ (empty symbols) and $\alpha=1.8$ (filled symbols) illustrate the results of Eq. (12). Studied values of N_0 are $N_0=10$ (○), 50 (◇), 100 (+). The slopes of fitted lines are 1.067 and 0.920, respectively, which corresponds to the predicted values $(2\alpha+1)/\alpha(1+\alpha)=1.067$ and 0.913.

$$\langle S_{N_0}(t)\rangle=\sum_{\mathbf{r}}\left[1-\left(\prod_{\tau=1}^{t}\Gamma_\tau(\mathbf{r})\right)^{N_0}\right], \tag{2}$$

where the sum is over all lattice sites. For the function $\Gamma_t(r)$ one has the following approximations [16]:

$$\Gamma_t(r)\approx\begin{cases}\exp\left(-K\dfrac{t^2}{|\mathbf{r}|^{1+\alpha}}\right), & \alpha<d \quad (3)\\ \exp\left(-L\dfrac{t^2}{\ln(t)|\mathbf{r}|^2}\right), & \alpha=1,\quad d=1 \quad (4)\\ \exp\left(-M\dfrac{t^{1+1/\alpha}}{|\mathbf{r}|^{1+\alpha}}\right), & \alpha>1,\quad d=1. \quad (5)\end{cases}$$

These approximations are valid for the large $|\mathbf{r}|$, $|\mathbf{r}|\gg t^{1/\alpha}$. However, as we shall see, it is enough to get the leading term of the asymptotic expansion of $\langle S_{N_0}(t)\rangle$. We restrict our attention to the region $|\mathbf{r}|\geqslant t^{1/\alpha}$ and denote by $\langle S_{N_0}(t)\rangle^+$ the average number of sites visited in this area. The analog of Eq. (2) reads

$$\langle S_{N_0}(t)\rangle^+=\sum_{|\mathbf{r}|\geqslant t^{1/\alpha}}\left[1-\left(\prod_{\tau=1}^{t}\Gamma_\tau(\mathbf{r})\right)^{N_0}\right]. \tag{6}$$

The number of sites visited inside the sphere $|\mathbf{r}|\leqslant t^{1/\alpha}$, denoted by $\langle S_{N_0}(t)\rangle^-$, is naturally bounded by the number of all sites in the sphere and will be shown to be asymptotically less than $\langle S_{N_0}(t)\rangle^+$.

II. THE ONE-DIMENSIONAL CASE

We approximate the summation in Eq. (2) by an integral and get

$$\langle S_{N_0}(t)\rangle^+\approx\int_{|\mathbf{r}|\geqslant t^{1/\alpha}}\left[1-\left(\prod_{\tau=1}^{t}\Gamma_\tau(\mathbf{r})\right)^{N_0}\right]d\mathbf{r}. \tag{7}$$

We begin with the case $\alpha<1$. Substitution of the expression from Eq. (3) and introduction of the change of variables $\mathbf{r}=(N_0t^3)^{1/(d+\alpha)}$ yield

$$\begin{aligned}\langle S_N(t)\rangle&\approx\int_{|\mathbf{r}|\geqslant t^{1/\alpha}}\left[1-\exp\left(-K'N_0\frac{t^3}{|\mathbf{r}|^{d+\alpha}}\right)\right]d\mathbf{r}\\&=(N_0t^3)^{d/(d+\alpha)}\\&\quad\times\int_{|\mathbf{s}|\geqslant\sigma(t)}[1-\exp(-K'|\mathbf{s}|^{-(d+\alpha)})]d\mathbf{r},\end{aligned} \tag{8}$$

TABLE I. Asymptotic results for different dimensions and α regimes.

	$\alpha<1/2$	$1/2\leqslant\alpha<1$	$\alpha=1$	$1<\alpha<3/2$	$3/2<\alpha<2$
$d=1$	N_0t^2	$(N_0t^3)^{1/(1+\alpha)}$	$(N_0t^3/\ln t)^{1/2}$	$N_0^{1/(1+\alpha)}t^{(2\alpha+1)/\alpha(1+\alpha)}$	
$d=2$		N_0t^2		$(N_0t^3)^{2/(2+\alpha)}$	
$d=3$			N_0t^2		$(N_0t^3)^{3/(3+\alpha)}$
$d\geqslant4$				N_0t^2	

TABLE II. Asymptotic results obtained in [16] for the number of distinct sites visited by N Lévy flights without injection, $\langle S_N(t)\rangle$.

	$\alpha<1$	$\alpha=1$	$1<\alpha<2$	$\alpha=2$	$\alpha>2$
$N\to\infty$	$(Nn^2)^{1/(1+\alpha)}$	$N^{1/2}n(\ln n)^{-1/2}$	$N^{1/(1+\alpha)}n^{1/\alpha}$	$N^{1/3}n^{1/2}$	$N^{1/(1+\alpha)}n^{1/\alpha}$
$n\to\infty$	Nn	$N^{1/2}n(\ln n)^{-1/2}$	$N^{1/(1+\alpha)}n^{1/\alpha}$	$N^{1/3}n^{1/2}(\ln n)^{-1}$	$(n\ln N)^{1/2}$

where $\sigma(t)=N_0^{-1/(d+\alpha)}t^{(d-2\alpha)/\alpha(d+\alpha)}$. One can note that as t increases to infinity three cases can be distinguished: $\sigma\to 0$ for $\alpha>1/2$, σ is fixed for $\alpha=1/2$, and $\sigma\to\infty$ for $\alpha<1/2$. In the first two cases, $\alpha\geqslant 1/2$, we immediately get

$$\langle S_{N_0}(t)\rangle^+\propto(N_0t^3)^{d/(d+\alpha)},\quad \alpha\geqslant 1/2. \tag{9}$$

In the last case, $\alpha<1/2$, we decompose the exponent in Eq. (8) as $\exp(-|\mathbf{s}|^{-d-\alpha})\approx 1-|\mathbf{s}|^{-d-\alpha}$ and explicit integration implies

$$\langle S_{N_0}(t)\rangle^+\propto N_0t^2,\quad \alpha<1/2. \tag{10}$$

In Figs. 1 and 2 we show that the results of Monte Carlo (MC) simulations for the cases $\alpha=0.4$, $\alpha=0.65$, and $\alpha=0.8$ are in a good agreement with the predictions of the Eq. (9) and Eq. (10).

Analogous analysis for the case $\alpha\geqslant 1$ yields

$$\langle S_{N_0}(t)\rangle^+\propto\begin{cases}\left(\dfrac{N_0t^3}{\ln(t)}\right)^{1/2}, & \alpha=1 \quad (11)\\ (N_0)^{1/(1+\alpha)}t^{(2\alpha+1)/\alpha(1+\alpha)}, & \alpha>1. \quad (12)\end{cases}$$

MC simulations presented in Fig. 3 support these results.

III. ASYMPTOTIC RESULTS FOR HIGHER DIMENSIONS

Equation (8) remains valid for higher dimensions for all values of α ($\alpha<2$). The same reasoning as in the preceding section leads to the following conclusions. The result, Eq. (9),

$$\langle S_{N_0}(t)\rangle^+\propto(N_0t^3)^{d/(d+\alpha)}$$

is valid for $d-2\alpha\leqslant 2$, i.e., in $d=2$ for $\alpha\geqslant 1$ and in $d=3$ for $\alpha\geqslant 3/2$. Approximation, Eq. (10),

$$\langle S_{N_0}(t)\rangle^+\propto Nt^2$$

is valid in $d=2$ for $\alpha<1$, in $d=3$ for $\alpha<2/3$ and in $d\geqslant 3$ for all values of α.

IV. FINAL RESULTS AND CONCLUDING REMARKS

As noted in Sec. I the average number of distinct sites visited inside the sphere $|\mathbf{r}|<t^{1/\alpha}$, $\langle S_{N_0}(t)\rangle^-$, is bounded by the number $Ct^{d/\alpha}$ of all sites in the sphere, where C is a dimension dependent constant. It is easy to see that for $\alpha>d/2$ the term $\langle S_{N_0}(t)\rangle^+$ is dominant in the sum

$$\langle S_{N_0}(t)\rangle=\langle S_{N_0}(t)\rangle^-+\langle S_{N_0}(t)\rangle^+.$$

Thus final results for $\langle S_{N_0}(t)\rangle$ are presented by the corresponding equations for $\langle S_{N_0}(t)\rangle^+$.

To get results for $\alpha<d/2$ we note that the maximum possible number of sites visited is $N_0t(t-1)/2$ (each particle finds a new site each time step) and this upper bound on the order of magnitude is already reached by the corresponding approximations for $\langle S_{N_0}(t)\rangle^+$. Thus it cannot be increased by adding the term for $\langle S_{N_0}(t)\rangle^-$.

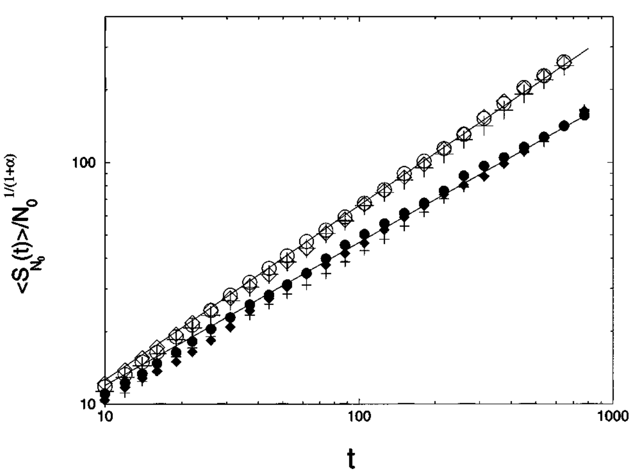

FIG. 4. Results of MC simulations for $\alpha=2.5$ (empty symbols) and $\alpha=3.0$ (filled symbols) support predictions of Eq. (13). The fitted slopes are 0.72 and 0.59, respectively, which are to be compared to the predictions $(2\alpha+1)/\alpha(1+\alpha)=0.685$ and 0.583.

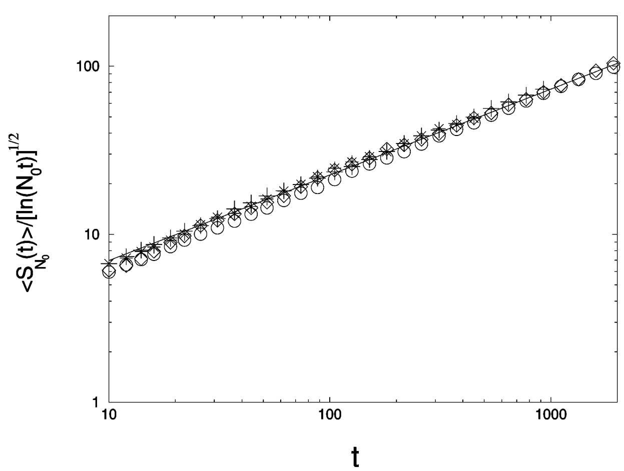

FIG. 5. Plot of the normalized function $\langle S_{N_0}(t)\rangle/\sqrt{\ln(N_0t)}$ supports prediction of Eq. (14). The slope of the fitted line is 0.51, which is to be compared to the prediction of 0.5.

We summarize all results of the present work in Table I. We discuss these results in detail for $d=1$, while extension of this discussion can be easily made.

It should be noted that the results in Table I have a functional form similar to those of [16], see Table II, subject to substitution $N=N_0t$, but to produce the present results in the limit $t\rightarrow\infty$ one has to make use of the results of Table II for both limits, $t\rightarrow\infty$ and $N\rightarrow\infty$. It is interesting to note that for $\alpha<1/2$ the functional form of $\langle S_{N_0}(t)\rangle$ coincides with that of $\langle S_N(t)\rangle$ in the limit $t\rightarrow\infty$, while for $1/2\leqslant\alpha<2$ the functional form for the limit $N\rightarrow\infty$ is used. To understand this behavior we recollect the condition of crossover between these two limits, $t\rightarrow\infty$ and $N\rightarrow\infty$ [Eq. (25) in [16]],

$$t\leqslant N^{\alpha/(1-\alpha)}.$$

To make use of this condition we substitute $N=N_0t$, as before, and get

$$t^{(1-2\alpha)/(1-\alpha)}\leqslant N_0^{\alpha/(1-\alpha)},$$

where N_0 is fixed. It is thus clear that in the limit $t\rightarrow\infty$ the inequality holds for $\alpha>1/2$, which explains the change in the functional form of $\langle S_{N_0}(t)\rangle$ at $\alpha=1/2$.

The substitution of the overall number of particles N_0t to the appropriate term produces correct results not only in the case of Lévy flights; in the case of Gaussian random walk with injection Berezhkovskii and Weiss [27] found analogous behavior. Thus it is natural to assume that substitution $N=N_0t$ leads to the correct results for the case $\alpha\geqslant2$, where the second moment becomes finite. Again we have to study the condition of crossover between $N\rightarrow\infty$ and $t\rightarrow\infty$ regimes for $\alpha\geqslant2$. This leads to the transcendental inequality

$$t^{(\alpha-2)/\alpha}\geqslant\frac{(N_0t)^{2/(1+\alpha)}}{\ln(N_0t)},$$

which produces the following asymptotic results.

The critical value of α is now the positive root of the equation $\alpha^2-3\alpha-2=0$, $\alpha_c=(3+\sqrt{17})/2$,

$$\langle S_{N_0}(t)\rangle=\begin{cases}N_0^{1/(1+\alpha)}t^{(2\alpha+1)/\alpha(1+\alpha)}, & 2\leqslant\alpha<\alpha_c \quad (13)\\ (t\ln(N_0t))^{1/2}, & \alpha>\alpha_c. \quad (14)\end{cases}$$

This result is supported by MC simulations for $\alpha<\alpha_c$, Fig. 4, and $\alpha>\alpha_c$, Fig. 5.

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