One - level trees:

Assume the following model for S_t: at $t=0$ S_0,
at $t=T$ either S_u (market up)
or S_d (market down)

Note: You do not know the probabilities of the market moves.

You ~~sold~~ want to find the fair value of a call with E: $S_d < E < S_u$, expiration T.

Idea: If you sell the call, you will lose money if the market is up. Compensate by investing: if the market is up you make profit on investing, if the market is down, you make profit on call premium.

Assume for now $r=0$

@ $t=0$ buy Δ shares (will determine Δ ...)
sell call

@ $t=T$ sell shares, settle call

$S_T = S_u$	$S_T = S_d$
$\Delta(S_u - S_0) + C - (S_u - E) = 0$	$\Delta(S_d - S_0) + C - 0 = 0$

$$\underbrace{\Delta(S_u - S_0)}_{\text{profit from stock}} + \underbrace{C}_{\text{call premium}} - \underbrace{(S_u - E)}_{\text{call payoff}} = 0$$

$$\underbrace{\Delta(S_d - S_0)}_{\text{loss from stock}} + \underbrace{C}_{\text{call premium}} - \underbrace{0}_{\text{call payoff}} = 0$$

<u>Note</u>; Really would like ≥ 0 in both but $= 0$ is the best possible if C is to be <u>fair</u>.

$$② \Rightarrow \quad C = (S_0 - S_d)\cdot\Delta$$

$$① - ② \Rightarrow \quad \Delta(S_u - S_d) = S_u - E$$

$$\therefore \quad C = (S_0 - S_u)\,\Delta \quad \text{where} \quad \Delta = \frac{S_u - E}{S_u - S_d}$$

One-level tree via replicating portfolio

Let us (a) put in $r > 0$ (b) recast the derivation in portfolio terms

Let $\Pi_1 = \{1 \text{ call } (E, T)\}$

$\Pi_2 = \{\Delta \text{ stock}, -y \text{ cash}\}$

If $\Pi_2(T) = \Pi_1(T)$, by Portfolio Lemma

$$\Pi_1(0) = \Pi_2(0)$$

want to find — can calculate

$t = T$	$S_T = S_u$	$S_T = S_d$
	$\Pi_2(T) = \Delta\cdot S_u - ye^{rT}$	$\Pi_2(T) = \Delta\cdot S_d - ye^{rT}$
	$= \Pi_1(T) = S_u - E$	$= \Pi_1(T) = 0$

$$\begin{cases} \Delta \cdot S_u - y e^{rT} = S_u - E \\ \Delta \cdot S_d - y e^{rT} = 0 \end{cases}$$

$$① - ② \Rightarrow \Delta \cdot (S_u - S_d) = S_u - E$$

$$\Delta = \frac{S_u - E}{S_u - S_d}, \quad y = e^{-rT} \cdot S_d \cdot \Delta$$

$$\Pi_1(0) = \Pi_2(0) = \Delta \cdot S_0 - y$$

$$= \Delta S_0 - \Delta \cdot e^{-rT} S_d$$

$$\therefore \quad C = \left(S_0 - e^{-rT} S_d\right) \frac{S_u - E}{S_u - S_d}$$

Example:

$S_0 = 100$ $\quad S_u = 110$ $\quad S_d = 90$ $\quad E = 105$

$r = 0.02 \left[\frac{1}{\text{year}}\right]$ $\quad T = 1$ year

(a) Price the call

$$C = \left(100 - e^{-0.02} 90\right) \frac{110 - 105}{110 - 90} \approx \$2.95$$

(after rounding)

(b) Suppose we sell 1 call for \$3.00

(this includes \$0.05 markup / commission)

Follow the Δ-hedging procedure

$$\Delta = \frac{110-105}{110-90} = \frac{5}{20} = \frac{1}{4} = 0.25$$

@ $t=0$ sell call

buy 0.25 stock (borrow from the bank)

Balance in the bank : $3.00 - 0.25 \times 100 = \$22$

@ $t=T$ settle call

sell stock

repay bank (with interest)

$S_T = 110$	$S_T = 90$
Balance $= -(110-105) + 0.25 \times 110 - 22 \times e^{0.02}$	Bal $= -0 + 0.25 \times 90 - 22 \times e^{0.02}$
≈ 0.06	≈ 0.06

H/W: Show that if more precision is used in all calculations, $(3.00 - C) \times e^{0.02} =$ Final Bal.

One level tree with a general payoff

Suppose we want to find the fair value of a derivative (call or put or their combination) whose payoff structure is given

V is the value at time $t=0$

$$\begin{tikzcd}[row sep=small] & S_u \\ S_0 \arrow[ur, dash] \arrow[dr, dash] & \\ & S_d \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & V_u \\ V \arrow[ur, dash] \arrow[dr, dash] & \\ & V_d \end{tikzcd}$$

Stock

want to find

given

<u>Examples:</u> Call : $V_u = S_u - E$, $V_d = 0$

Put : $V_u = 0$ $\quad V_d = E - S_d$

Create a replicating portfolio $\Pi_2 = \{\Delta \text{ stock}, -y \text{ cash}\}$

"Replicating" conditions : $\begin{cases} \Delta \cdot S_u - y e^{rT} = V_u \\ \Delta \cdot S_d - y e^{rT} = V_d \end{cases}$

$$\Delta = \frac{V_u - V_d}{S_u - S_d} \qquad y = e^{-rT}\left(\Delta \cdot S_d - V_d\right)$$

Then
$$\begin{aligned} V = \Pi_2(0) &= \Delta \cdot S_0 - y \\ &= e^{-rT}\left(e^{rT} S_0 \cdot \Delta - S_d \cdot \Delta + V_d \right) \\ &= e^{-rT}\left((e^{rT} S_0 - S_d) \frac{V_u - V_d}{S_u - S_d} + V_d \right) \\ &= e^{-rT}\left(V_u \cdot \frac{e^{rT} S_0 - S_d}{S_u - S_d} + V_d \cdot \left(1 - \frac{e^{rT} S_0 - S_d}{S_u - S_d}\right)\right) \\ &= \text{P.V.}\left(V_u \cdot q + V_d \cdot (1-q) \right), \quad q = \frac{e^{rT} S_0 - S_d}{S_u - S_d} \end{aligned}$$

Remarks: 1) We have put V in the form of an "expectation of a random variable" (V_u with probability q ; V_d with probability $1-q$) converted to present value

2) This "probability" q is an artificial construct. It is not reflecting any real life probabilities (if they exist!). But it makes the market appear to be a "fair lottery":

$$\mathbb{E}_q S_T = S_u \cdot q + S_d (1-q) = e^{rT} \cdot S_0$$ Martingale Property

H/W: Prove the latter equality

3) If we define $S_u = u \cdot S_0$, $S_d = d \cdot S_0$

$$V = P.V.\left(V_u \cdot q + V_d \cdot (1-q) \right) \qquad q = \frac{e^{rT} - d}{u - d}$$

4) q is called "risk-neutral probability".