Distribution of a function of a random variable

Let X be a r.v. with density $f_X(x)$.

What is the density of $Y = \varphi(X)$?

From definition of density,

$$f_Y(y) = \frac{d}{dy}\mathbb{P}(Y \le y) = \frac{d}{dy}\mathbb{P}(Y \in (a, y))$$

for any fixed a.

Assume φ is monotone increasing

$$\mathbb{P}(a \le Y \le y) = \mathbb{P}(a \le \varphi(X) \le y)$$

$$= \mathbb{P}(\varphi^{-1}(a) \le X \le \varphi^{-1}(y))$$

$$= \int_{\varphi^{-1}(a)}^{\varphi^{-1}(y)} f_X(x)\, dx$$

Change variables (u-sub) $u = \varphi(x)$

(We can also differentiate this using Leibniz Formula

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x,y)\, dx = f(b(y), y)\, b'(y) - f(a(y), y)\, a'(y) + \int_{a(y)}^{b(y)} \frac{\partial f}{\partial y}\, dx$$

)

$$du = \varphi'(x)\, dx \qquad \text{or} \qquad x = \varphi^{-1}(u) \quad \text{and}$$

$$dx = \left(\varphi^{-1}(u)\right)' du$$

$$x = \varphi^{-1}(a) \;..\; \varphi^{-1}(y)$$

$$u = a \;..\; y$$

$$\mathbb{P}(a \le Y \le y) = \int_a^y f_X(\varphi^{-1}(u)) \left(\varphi^{-1}(u)\right)' du$$

Therefore, $f_Y(y) = f_X(\varphi^{-1}(y)) \left(\varphi^{-1}(y)\right)'$.

If φ is monotone decreasing,

$$\mathbb{P}(a \le \varphi(X) \le y) = \mathbb{P}(\varphi^{-1}(y) \le X \le \varphi^{-1}(a))$$

$$= \ldots = \int_y^a f_X(\varphi^{-1}(u)) \left(\varphi^{-1}(u)\right)' du$$

Note that in this case also $\left(\varphi^{-1}(u)\right)' \le 0$

$$= \int_a^y f_X(\varphi^{-1}(u)) \left|\left(\varphi^{-1}(u)\right)'\right| du$$

Therefore, for monotone φ

$$f_Y(y) = f_X(\varphi^{-1}(y)) \left|\frac{d}{dy}\varphi^{-1}(y)\right|.$$

If φ is not monotone, there is also a general formula (summing over preimages $\varphi^{-1}(y)$ of y), but it gets easier to derive the answer "by hand".

Example 1: $\quad X \sim N(0,1)$, $\quad \varphi(X) = X^2$

$Y = X^2 \geq 0$ so $f_Y(y) \equiv 0 \quad y < 0$

Let $y \geq 0$

$$f_Y(y) = \frac{d}{dy} \mathbb{P}(0 \leq X^2 \leq y)$$

$$= \frac{d}{dy} \mathbb{P}(|X| \leq \sqrt{y})$$

$$= \frac{d}{dy} \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

by Leibniz Rule (or via u-sub $u = x^2$)

$$= \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \cdot (\sqrt{y})' - \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2} \cdot (-\sqrt{y})'$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-y/2}.$$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y/2} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

(density is ∞ at $y = 0$, but integrable, so it's OK)

Example 2: $X \sim U(0,1)$ uniform distr. on $(0,1)$

$$Y = \varphi(X) = -\frac{1}{\lambda} \ln X, \quad \lambda > 0$$

Since $\ln X < 0$, $\lambda > 0$, we have $Y > 0$

so $f_Y(y) = 0$ if $y \leq 0$

φ is monotone (decreasing)

φ^{-1}: $Y = -\frac{1}{\lambda} \ln X$

$$\ln X = -\lambda Y$$

$$X = e^{-\lambda y}$$

$\varphi^{-1}(y) = e^{-\lambda y}$ $\quad \left(\varphi^{-1}(y)\right)' = -\lambda e^{-\lambda y}$

$$f_X(x) = 1 \quad x \in (0,1)$$

Note that $\varphi^{-1}(y) \in (0,1)$ for all $y > 0$.

$\Rightarrow$ $f_Y(y) = 1 \cdot \left| -\lambda e^{-\lambda y} \right| = \lambda e^{-\lambda y}$

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y > 0 \\ 0 & y \leq 0 \end{cases}$$

This is exponential distribution.