Example 3 :
$$X \sim N(0,1)$$
$$Y = \varphi(X) = e^{m+sX}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \varphi^{-1}(y) = \frac{\ln y - m}{s}$$

$\varphi$ monotone $\mathbb{R} \to (0,\infty)$

$$\Rightarrow \quad f_Y(y) = \begin{cases} \frac{1}{sy} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln y - m)^2}{2s^2}} & y > 0 \\ 0 & y \leq 0 \end{cases}$$

Further useful questions :

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{m+sx} \, dx$$

$$\left( \text{since} \quad \mathbb{E}[Y] = \int y f_Y(y)\, dy = \underbrace{\int \varphi(x) f_X(x)\, dx}_{\text{easier}} \right)$$

$$= e^m \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2sx + s^2 - s^2)} \, dx$$

$$= e^{m+\frac{1}{2}s^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-s)^2} \, dx = e^{m+\frac{1}{2}s^2}$$

In particular, if $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t} Z}$

$$Z \sim N(0,1)$$

here $(\mu - \frac{1}{2}\sigma^2)t = m \qquad s = \sigma\sqrt{t}$, so

$$\mathbb{E}[S_t] = e^{m + s^2/2} = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma^2 t/2} = e^{\mu t}$$

Coming back to $Y$,

$$\mathbb{E}[Y^2] = \mathbb{E}\left[e^{2m + 2sX}\right]$$

$$= \mathbb{E}\left[e^{\tilde{m} + \tilde{s}X}\right] \qquad \begin{array}{l} \tilde{m} = 2m \\ \tilde{s} = 2s \end{array}$$

$$= e^{\tilde{m} + \tilde{s}^2/2} = e^{2(m + s^2)}$$