Itō Formula

$f = f(t, X)$ smooth

Then
$$df(t, W_t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right) dt + \frac{\partial f}{\partial W} dW_t$$

(using " $dW_t^2 = dt$ ")

Itō Chain Rule

$f = f(t, X)$ $\quad$ $g = g(t, X)$

$$df(t, g(t, W_t)) = \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial g^2}(dg)^2 + \frac{\partial f}{\partial g} dg$$

by Itō formula $\quad dg = \left(\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial W^2}\right) dt + \frac{\partial g}{\partial w} dW_t$

using $\quad dW_t^2 = dt \quad dt^2 = 0 \quad dW_t \cdot dt = 0$

$$dg^2 = \left(\frac{\partial g}{\partial w}\right)^2 dt$$

$$df(t, g(t, W_t)) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial g^2}\left(\frac{\partial g}{\partial W}\right)^2\right) dt + \frac{\partial f}{\partial g} dg(t, W_t)$$

Examples of Itō differential

Ex 1: Calculate $d(W_t^2)$.

Here $f(t, W) = W^2$, so $\frac{\partial f}{\partial t} = 0$ $\frac{\partial f}{\partial w} = 2w$

$\frac{\partial^2 f}{\partial W^2} = 2$, therefore, by Itō formula

$$d(W_t^2) = dt + 2 W_t \, dW_t$$

Ex 2: Our continuous-time asset price model is $S_t = S(t, W_t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$

$\frac{\partial S}{\partial t} = (\mu - \frac{\sigma^2}{2}) S_t$, $\frac{\partial S}{\partial w} = \sigma S_t$, $\frac{\partial^2 S}{\partial W^2} = \sigma^2 S_t$

By Itō formula

$$d S(t, W_t) = \mu S_t \, dt + \sigma S \, dW_t$$

This is Stochastic Differential Equation of "Geometric Brownian Motion"

Ex 3: A "financial derivative" derives its value from the value of underlying stock $S_t = S(t, W_t)$. Its value also depends on time. Therefore, $V = V(t, S_t)$, where $dS_t = \mu S_t dt + \sigma S_t dW_t$

Calculate $dV(t, S_t)$.

By Itō Chain Rule, (using $(dS_t)^2 = \sigma^2 S^2 dt$)

$$dV(t, S_t) = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial V}{\partial S} dS_t$$

<u>Ex 4:</u> Let $dX_t = \mu X_t dt + \sigma X_t dW_t$

Calculate dY_t, where $Y_t = \frac{1}{X_t}$.

Use Itō Chain Rule with

$$f(t, g) = \frac{1}{g}, \qquad g = X_t$$

Then $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial g} = -\frac{1}{g^2}$, $\frac{\partial^2 f}{\partial g^2} = \frac{2}{g^3}$

$$dY_t = df(t, g_t) = 0 + \frac{1}{2}\frac{2}{g^3}(dg)^2 + \left(-\frac{1}{g^2}\right) dg$$

subst. $g = X_t$, $dX_t = \mu X_t dt + \sigma X_t dW_t$

$$(dX_t)^2 = \sigma^2 X_t^2 dt$$

$$dY_t = \frac{1}{X_t^3}\sigma^2 X_t^2 dt - \frac{1}{X_t^2}\left(\mu X_t dt + \sigma X_t dW_t\right)$$

$$= \frac{1}{X_t} \sigma^2 dt - \frac{1}{X_t} \mu dt - \frac{1}{X_t} \sigma dW_t$$

but $\frac{1}{X_t} = Y_t$, therefore

$$dY_t = (\sigma^2 - \mu) Y_t dt - \sigma Y_t dW_t$$

Related: Itō Product Rule

$$X = X(t, W_t) \qquad Y = Y(t, W_t)$$

(driven by the same Wiener process W_t)

Then
$$d(XY) = dX \cdot Y + X \cdot dY + dX \cdot dY$$

this term is too small in usual calculus but may contribute in stoch. calculus since $(dW_t)^2 = dt$