TEXAS A&M UNIVERSITY
DEPARTMENT OF MATHEMATICS

MATH 425

Midterm test version A, 19 Oct 2023

On my honor, as an Aggie, I have neither given nor received unauthorized aid on this work.

Name (print):

No detailed analytical work — no points.

Q.1 (7 points) Assuming $r = 0$, plot the profit (PNL) diagram of the spread consisting of a long put with strike $E = 105$ and premium 5.50 and 3 short puts with strike $E = 95$ and premium 1.20. Identify the range of the prices of the underlying where the spread will yield profit.

Q.2 (15 points, use two pages) Compute the 3-level binomial tree for a European Call with the following parameters: $S_0 = 100$, $E = 90$, $r = 0.05$, $T = 3$ months (i.e. 3/12 of a year) using $u = 1.1$, $d = 0.9$. Separately, compute the 3-level binomial tree for a European Put with identical parameters.

Compute Delta in both trees at the nodes $S = 100$, $S = 90$ and $S = 99$ (i.e. the path Down, Up). Comparing Call Delta and Put Delta at equivalent nodes, conjecture the mathematical relationship between them. Prove this relationship using Put-Call parity.

Q.3 (8 points) Let U be a uniform random variable on the interval $(0, 1)$ (density $p(u) = 1$ if $u \in [0, 1]$ and $p(u) = 0$ otherwise). Calculate the support (the set of values the random variabe can take) and the probability density function of the random variable $X = -\ln(U)/\lambda$, where $\lambda > 0$. Hint: calculate the cumulative distribution function $\mathbb{P}[X \leq x]$, then differentiate it with respect to x.

Formulas that might be useful (do not detach)

- Call and put payoff

$$C_T(S_T, E) = \max(0, S_T - E), \qquad P_T(S_T, E) = \max(0, E - S_T).$$

- Put-call parity and price bounds

$$C_t(S, E) + Ee^{-r(T-t)} = P_t(S, E) + S_t,$$
$$\max(S_t - Ee^{-r(T-t)}, 0) \le C_t(S_t, E) \le S_t,$$
$$\max(Ee^{-r(T-t)} - S_t, 0) \le P_t(S_t, E) \le Ee^{-r(T-t)}.$$

- Price of the forward contract

$$F_t = S_t e^{r(T-t)}.$$

- Tree back-propagation formula

$$V^{\text{next}} = e^{-r\Delta t}\left(V_u^{\text{prev}} q + V_d^{\text{prev}}(1-q)\right), \qquad \text{where} \quad q = \frac{e^{r\Delta t} - d}{u - d}.$$

- Tree Delta

$$\Delta = \frac{V_u - V_d}{S_u - S_d}.$$

- Short-cut formula for European-type claims

$$V_0^0 = e^{-rT} \sum_{j=0}^{L} V_j^L \binom{L}{j} q^j (1-q)^{L-j}, \quad q = \frac{e^{r\Delta t} - d}{u - d},$$

where the price of the underlying is assumed to be $S_j^L = S_0 u^j d^{L-j}$.
- Expectation is linear:

$$\mathbb{E}(\alpha_1 X_1 + \alpha_2 X_2 + \beta) = \alpha_1 \mathbb{E} X_1 + \alpha_2 \mathbb{E} X_2 + \beta.$$

- Probability density function of random variable X is an integrable function $f_X(x)$ such that

$$F_X(x) := \mathbb{P}(X < x) = \int_{-\infty}^{x} f_X(s)ds.$$

In other words (when $f_X(x)$ is continuous)

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \mathbb{P}(X < x).$$

- Properties of the variance:

$$\text{var}(\alpha X + \beta) = \alpha^2 \, \text{var}(X),$$
$$\text{var}(X_1 + X_2) = \text{var}(X_1) + \text{var}(X_2) \qquad \text{if } X_1, X_2 \text{ are uncorrelated.}$$

- Probability density function of a normal $N(\mu, \sigma^2)$ variable is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

- If $Z \sim N(0,1)$ then $X = \sigma Z + \mu$ has distribution $N(\mu, \sigma^2)$. In particular, normal variables remain normal under linear transformations $X \mapsto \alpha X + \beta$.