TEXAS A&M UNIVERSITY
DEPARTMENT OF MATHEMATICS

MATH 425-500

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On my honor, as an Aggie, I have neither given nor received unauthorized aid on this work.

Name (print):

- You **must** show work. No work no points.
- Each question is worth 10 points. Solve 4 questions, **get one free**. You may attempt and attach all five questions but to get free points you **must** clearly mark which question is **"Free Pass"**. If I grade it, you get what you earned, not the full points.

1. Show using Portfolio Lemma that the Call price $C_t(S, E, T)$ is a decreasing function of the strike E (assuming all other parameters are kept constant). To be more precise, show that if $E_2 \geq E_1$, then $C_t(S, E_2, T) \leq C_t(S, E_1, T)$. You may use payoff diagrams as a part of your argument.

2. Let $S = S(t, W_t)$ satisfy the stochastic equation

$$dS = 0.075 S dt + 0.3 S dW.$$

Use Ito chain rule

$$df\big(t, g(t, W_t)\big) = \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial g^2} (dg_t)^2 + \frac{\partial f}{\partial g} dg_t,$$

to find the stochastic differential $d(\ln S)$.

3. Price the following *American put option* using the tree model: stock price now is $S_0 = 100$, strike is $E = 110$, interest rate is 5%, time to expiration 3 months.

1. Construct the 3-level tree (1 level per month) if the stock price is expected to either go up by 10% or go down by 10%. Price the above option using this tree.
2. Describe the hedging procedure undertaken by a writer of the option who seeks to eliminate risk. You must assume that the holder of the option will do what is best for them (i.e. exercise early if optimal). Consider the price path: Down, Up, Up.

You may use 2 pages for your answer.

4. Consider Black–Scholes–Merton formula for the call price $C(t, S, E, \sigma, T)$ as a function of strike E and calculate the derivative $\frac{\partial C}{\partial E}$. What can you say about its sign?
(Hint: Compare your answer about the sign to Question 1.)

5. By taking the limit $N \to \infty$ and $\Delta t \to 0$ ($T = N\Delta t$ is fixed) in the tree evaluation, we get the formula for the value of an option with the given payoff

$$V = e^{-rT} \int_{-\infty}^{\infty} \text{Payoff}\left(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}z}\right) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz. \tag{1}$$

Use this formula to calculate the value of stock-or-nothing option,

$$\text{Payoff}(S) = \begin{cases} S & \text{if } S > E, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

You may use 2 pages for your derivation.

Points: /50

Formulas that might be useful (detach and take with you)

- Call / put payoff

$$C_T(S_T, E) = \max(0, S_T - E), \qquad P_T(S_T, E) = \max(0, E - S_T).$$

- Put-call parity

$$C_t(S, E) + Ee^{-r(T-t)} = P_t(S, E) + S_t.$$

- Probability density function of random variable X is integrable function $f_X(x)$ such that

$$F_X(x) := \mathbb{P}(X < x) = \int_{-\infty}^{x} f_X(s)ds.$$

In other words (when $f_X(x)$ is continuous)

$$f_X(x) = \frac{d}{dx} F_X(x).$$

- Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy.$$

- Tree back-propagation formula

$$V^{next} = e^{-r\delta t}\left(V_{up}^{prev} q + V_{down}^{prev}(1-q)\right), \quad \text{where} \quad q = \frac{e^{r\delta t} - d}{u - d}.$$

- Tree Delta

$$\Delta = \frac{V_{up} - V_{down}}{S_{up} - S_{down}}.$$

- Black–Scholes–Merton model for call / put price

$$C(t, S, E) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \qquad P(t, S, E) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where

$$d_{1,2} = \frac{\ln(S/E) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

- useful idenitity for deriving the Greeks

$$SN'(d_1) - Ee^{-r(T-t)}N'(d_2) = 0.$$

- Greeks for the call / put

	Call	Put
$\Delta = \frac{\partial V}{\partial S}$	$N(d_1)$	$-N(-d_1)$
$\Gamma = \frac{\partial^2 V}{\partial S^2}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$
vega= $\frac{\partial V}{\partial \sigma}$	$SN'(d_1)\sqrt{T-t}$	$SN'(d_1)\sqrt{T-t}$
$\theta = \frac{\partial V}{\partial t}$	$\frac{-S\sigma}{2\sqrt{T-t}}N'(d_1) - rEe^{-r(T-t)}N(d_2)$	$\frac{-S\sigma}{2\sqrt{T-t}}N'(d_1) + rEe^{-r(T-t)}N(d_2)$
$\rho = \frac{\partial V}{\partial r}$	$(T-t)Ee^{-r(T-t)}N(d_2)$	$-(T-t)Ee^{-r(T-t)}N(-d_2)$