TEXAS A&M UNIVERSITY
DEPARTMENT OF MATHEMATICS

MATH 425

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On my honor, as an Aggie, I have neither given nor received unauthorized aid on this work.

Name (print):

- You **must** show work. No work no points no kidding no cry.
- Each question is worth 10 points, except the binomial tree calculation and hedging, which is worth 20. You may use 2 pages for each question.

1. You observe the following call option prices: \$32.65 for $E = 170$ call and \$17.80 for $E = 190$ call. Assume $r = 0$.

1. Draw the profit diagram for the following spread (called "bull spread"): you buy $E = 170$ call and sell $E = 190$ call. Identify the price range in which you make a profit.
2. In our homework, we used No Arbitrage principle to establish the following inequality: assuming $E_1 < E_2$,
$$0 \le C(E_1) - C(E_2) \le E_2 - E_1.$$
Use this inequality[a] to bound possible prices of a call with the strike $E = 180$.

[a] Pricing a butterfly (or using convexity of $C(E)$) produces a better upper bound but takes more work.

2. (20 points) Price the following *European call option* using the tree model: stock price now is $S_0 = 100$, strike is $E = 95$, interest rate is 5%, time to expiration 3 months.

1. Construct the **3-level** tree (1 level per month) if the stock price is expected to either go up by 10% or go down by 10% in every month. Price the above option using this tree.
2. Describe the hedging procedure undertaken by a writer of the option who seeks to eliminate risk. Consider the price path: Down, Up, Up.
3. Calculate the total value of your portfolio at every point of the process. The total value must take into account prices of the option, stock and cash holdings. At time steps 1 and 2 calculate the portfolio value both before and after re-hedging. Please provide details of your calculation, not just write down the answers.
4. Explain why are the answers you got are to be expected.

3.

1. The Taylor expansion in two variables is

$$f(t+\Delta t, s+\Delta s) = f(t,s) + \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial s}\Delta s + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\Delta t^2 + \frac{\partial^2 f}{\partial t \partial s}\Delta t \Delta s + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\Delta s^2 + \dots.$$

   Obtain from it the following approximation for the price increment of a call

$$C(t+\Delta t, S+\Delta S) - C(t,S) = \Delta \cdot \Delta S + \Theta \Delta t + \frac{1}{2}\Gamma(\Delta S)^2 + o(\Delta t), \qquad (1)$$

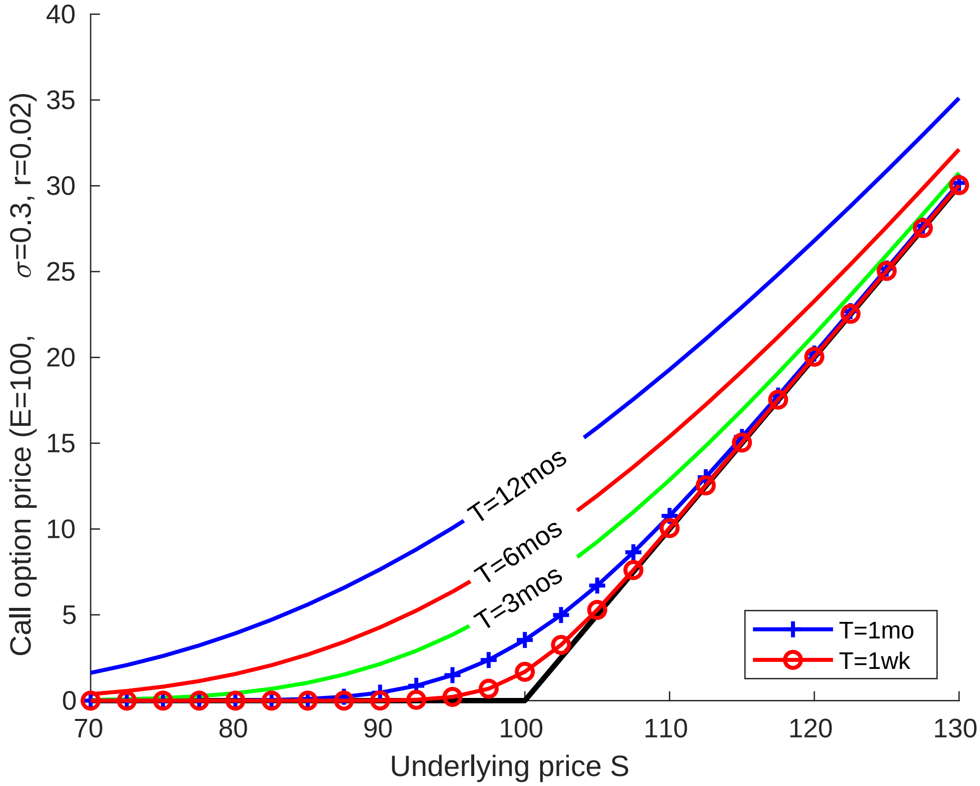
   where $o(\Delta t)$ stands for "terms smaller than Δt" (you may use $\Delta S \sim \sqrt{\Delta t}$ in your estimates).
2. At time $t = 0$ the stock price is $S_0 = 35.42$. You calculated (using Black–Scholes formula with parameters $E = 32.5$, $r = 0.03$, $\tau = 7/52$, $\sigma = 0.2$) the call price and the Greeks: $C = 3.18$, $\Delta = 0.8969$, $\Theta = -2.5897$, $\Gamma = 0.0690$. Use equation (1) to find an approximation to call price after one week when the stock price moved to $S_1 = 38$.

Remark: using Black–Scholes formula, the call price after one week is \$5.62.

4. We calculated the value of Δ for a call to be

$$\Delta = N(d_1), \qquad d_1 = \frac{\ln(S/E) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Find the limit of Δ as time approaches expiration: $t \to T$. Show work! Explain how your answer agrees with the following plot of option prices.



Points:	**/50**

Formulas that might be useful (do not detach)

- Call / put payoff
$$C_T(S_T, E) = \max(0, S_T - E), \qquad P_T(S_T, E) = \max(0, E - S_T).$$
- Put-call parity
$$C_t(S, E) + Ee^{-r(T-t)} = P_t(S, E) + S_t.$$
- Probability density function of random variable X is integrable function $f_X(x)$ such that
$$F_X(x) := \mathbb{P}(X < x) = \int_{-\infty}^{x} f_X(s)ds.$$
In other words (when $f_X(x)$ is continuous)
$$f_X(x) = \frac{d}{dx}F_X(x).$$
- Standard normal distribution
$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\, dy.$$
- Tree back-propagation formula
$$V^{next} = e^{-r\delta t}\left(V_{up}^{prev}q + V_{down}^{prev}(1-q)\right), \quad \text{where} \quad q = \frac{e^{r\delta t} - d}{u - d}.$$
- Tree Delta
$$\Delta = \frac{V_{up} - V_{down}}{S_{up} - S_{down}}.$$
- Black–Scholes–Merton model for call / put price
$$C(t, S, E) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \qquad P(t, S, E) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1),$$
where
$$d_{1,2} = \frac{\ln(S/E) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$
- useful identity for deriving the Greeks
$$SN'(d_1) - Ee^{-r(T-t)}N'(d_2) = 0.$$
- Greeks for the call / put

	Call	Put
$\Delta = \frac{\partial V}{\partial S}$	$N(d_1)$	$-N(-d_1)$
$\Gamma = \frac{\partial^2 V}{\partial S^2}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$
vega$= \frac{\partial V}{\partial \sigma}$	$SN'(d_1)\sqrt{T-t}$	$SN'(d_1)\sqrt{T-t}$
$\theta = \frac{\partial V}{\partial t}$	$\frac{-S\sigma}{2\sqrt{T-t}}N'(d_1) - rEe^{-r(T-t)}N(d_2)$	$\frac{-S\sigma}{2\sqrt{T-t}}N'(d_1) + rEe^{-r(T-t)}N(d_2)$
$\rho = \frac{\partial V}{\partial r}$	$(T-t)Ee^{-r(T-t)}N(d_2)$	$-(T-t)Ee^{-r(T-t)}N(-d_2)$