MATH 425 Fall 2021 Berkolaiko Final A

Must show work, write clearly with **dark pencil or pen**, each question on a separate page, submit in Gradescope, assign pages.

Q.1 (10 points) The following option prices (E is the strike) are given:

	$E = 320$	$E = 330$	$E = 340$
Call	16.00	8.60	3.80
Put	2.20	4.80	10.00

For a spread containing:
1 long 320 Call, 2 short 330 Calls and 1 long 340 Call,

(1) Draw the payoff diagram.
(2) Draw the profit diagram.
(3) Create a spread with identical payoff using Put options only. Explain why the setup cost of this spread must be the same. Check, with the prices provided, that it is.

Q.2 (20 points) Price the following *American put option* using the tree model: stock price now is $S_0 = 100$, strike is $E = 110$, interest rate is 5%, time to expiration 3 months.

(1) Construct the **3-level** tree (1 level per month) if the stock price is expected to either go up by 10% or go down by 10% in every month. Price the option using this tree.
(2) Describe the hedging procedure undertaken by a writer of the option who seeks to eliminate risk. Consider the price path: Up, Down, Down. Assume that the holder of the option will do what is best for them (i.e. exercise early if optimal).
(3) Calculate the writer's balance if the holder *does not* exercise when it's optimal, carrying the option to expiration instead? Explain your answer from the financial perspective.

Q.3 (10 points) If $X = X(t, W_t)$ satisfies the stochastic equation

$$dX = 0.05dt + 0.2dW_t,$$

find the equation satisfied by $S = e^{X-0.01t}$.

Q.4 (10 points) Recall the definition of the covariance,

$$\operatorname{cov}(X, Y) = \mathbb{E}\big[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\big] = \operatorname{cov}(Y, X).$$

(1) Show that $\operatorname{cov}(X, \alpha Y) = \alpha \operatorname{cov}(X, Y)$.
(2) Show that $\operatorname{cov}(X, Y_1 + Y_2) = \operatorname{cov}(X, Y_1) + \operatorname{cov}(X, Y_2)$.
(3) Let Z_1 and Z_2 be independent standard normal variables. Find the covariance of $X = 2Z_1 - Z_2$ and $Y = Z_1 - 3Z_2$.

Formulas that might be useful (do not detach)

- Call / put payoff
$$C_T(S_T, E) = \max(0, S_T - E), \qquad P_T(S_T, E) = \max(0, E - S_T).$$
- Put-call parity
$$C_t(S, E) + Ee^{-r(T-t)} = P_t(S, E) + S_t.$$
- Probability density function of random variable X is integrable function $f_X(x)$ such that
$$F_X(x) := \mathbb{P}(X < x) = \int_{-\infty}^{x} f_X(s)ds.$$
In other words (when $f_X(x)$ is continuous)
$$f_X(x) = \frac{d}{dx}F_X(x).$$
- Standard normal distribution
$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\, dy.$$
- Tree back-propagation formula
$$V^{next} = e^{-r\delta t}\left(V_{up}^{prev}q + V_{down}^{prev}(1-q)\right), \quad \text{where} \quad q = \frac{e^{r\delta t} - d}{u - d}.$$
- Tree Delta
$$\Delta = \frac{V_{up} - V_{down}}{S_{up} - S_{down}}.$$
- Ito formula: for $f = f(t, W)$
$$df_t = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}\right)dt + \frac{\partial f}{\partial w}dW_t.$$
- Ito chain rule: let $f = f(t, g)$, $g = g(t, w)$, then the differential of $f_t = f\big(t, g(t, W_t)\big)$ with respect to time is
$$df_t = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial g^2}(dg_t)^2 + \frac{\partial f}{\partial g}dg_t,$$
- Black–Scholes–Merton model for call / put price
$$C(t, S, E) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$
$$P(t, S, E) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1),$$
where
$$d_{1,2} = \frac{\ln(S/E) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$