TEXAS A&M UNIVERSITY
DEPARTMENT OF MATHEMATICS

MATH 425

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On my honor, as an Aggie, I have neither given nor received unauthorized aid on this work.

Name (print):

- You **must** show work. No work no point.
- Each question is worth 10 points, except the binomial tree calculation and hedging, which is worth 20.

1. (Can use 2 pages) You observe the following call option prices: \$32.64 for $E = 170$ call, \$23.35 for $E = 180$ call, and \$17.80 for $E = 190$ call. Assume $r = 0$.

1. Draw *both* the payoff and the profit diagrams for the following spread (called "butterfly"): you buy one $E = 170$ call and one $E = 190$ call and sell two $E = 180$ calls. Identify the price range in which you make a profit.
2. Suppose the price of the $E = 180$ call was not known. Using the other two prices and the payoff diagram you drew find the maximum price for this option that is consistent with No Arbitrage Principle.

2. (20 points, can use 2 pages) Price the following *American call option* using the tree model: stock price now is $S_0 = 100$, strike is $E = 90$, interest rate $r = 0.10$, time to expiration 3 months.

1. Construct the **3-level** tree (1 level per month) assuming the stock price goes up by 10% or goes down by 10% in every month. Price the above option using this tree.
2. Check that at every internal node of the tree, the recursively calculated option price is equal or larger than the early exercise payoff.
3. Describe the hedging procedure undertaken by the writer of the option who seeks to eliminate risk. Consider the price path: Down, Up, Up.

3. Your model for log-returns over the first hour of exchange trading is

$$\ln\left(\frac{S_h}{S_0}\right) = 0.08h + 0.15\sqrt{h}Z_1,$$

where Z_1 is standard normal. Your model for the log-returns over the second hour is

$$\ln\left(\frac{S_{2h}}{S_h}\right) = 0.1h + 0.08\sqrt{h}Z_2,$$

where Z_2 is standard normal and independent of Z_1.

What is the mean and the standard deviation of the 2-hour log-return $\ln\left(\frac{S_{2h}}{S_0}\right)$?

4. Your model for S_t satisfied the stochastic differential equation

$$dS_t = 0.08S_t dt + 0.2S_t dW_t. \tag{1}$$

Find the equation satisfied by $Y_t = S_t^2 e^{-0.03t}$. Hint: Find the differential of Y_t; the final equation should not contain any S_t.

Points: /50

Formulas that might be useful (do not detach)

- Call / put payoff

$$C_T(S_T, E) = \max(0, S_T - E), \qquad P_T(S_T, E) = \max(0, E - S_T).$$

- Put-call parity

$$C_t(S, E) + Ee^{-r(T-t)} = P_t(S, E) + S_t.$$

- Standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy.$$

- Properties of the normal

$$\begin{aligned} &Z \sim N(0,1) \quad \Rightarrow \quad a + bZ \sim N(a, b^2), \\ &X_1 \sim N(\mu_1, \sigma_1^2),\ X_2 \sim N(\mu_2, \sigma_2^2) \quad \Rightarrow \quad X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2). \end{aligned}$$

- Tree back-propagation formula

$$V^{next} = e^{-r\Delta t} \left(V_{up}^{prev} q + V_{down}^{prev}(1-q) \right), \quad \text{where} \quad q = \frac{e^{r\Delta t} - d}{u - d}.$$

- Tree Delta

$$\Delta = \frac{V_{up} - V_{down}}{S_{up} - S_{down}}.$$

- Ito formula: for $f = f(t, W)$

$$df_t = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial w^2} \right) dt + \frac{\partial f}{\partial w} dW_t.$$

- Ito chain rule: let $f = f(t, g)$, $g = g(t, w)$, then the differential of $f_t = f\big(t, g(t, W_t)\big)$ with respect to time is

$$df_t = \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial g^2} (dg_t)^2 + \frac{\partial f}{\partial g} dg_t,$$

- Black–Scholes–Merton model for call / put price

$$C(t, S, E) = SN(d_1) - Ee^{-r(T-t)} N(d_2), \qquad P(t, S, E) = Ee^{-r(T-t)} N(-d_2) - SN(-d_1),$$

  where

$$d_{1,2} = \frac{\ln(S/E) + \left(r \pm \frac{\sigma^2}{2} \right)(T - t)}{\sigma \sqrt{T - t}}.$$

- useful identity for deriving the Greeks

$$SN'(d_1) - Ee^{-r(T-t)} N'(d_2) = 0.$$