Solution of bonus problem 1

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Theorem 1.

$$\limsup_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

There are many different (equivalent!) definitions of upper limit — also known as limit superior.

Definition 1. Upper limit of x_n is the largest number L such that there is a subsequence of x_n converging to L.

Definition 2. Upper limit of x_n is such a number L that for any $\varepsilon > 0$ the inequality $x_n > L + \varepsilon$ is satisfied **finitely** many times and the inequality $x_n > L - \varepsilon$ is satisfied **infinitely** often.

Definition 3. Upper limit of x_n is

$$L = \lim_{k\to\infty}\left(\sup_{n>k} x_n\right).$$

If we allow limits to be equal to $\pm\infty$ then each sequence will have an upper limit (not so for limit). Also, if the limit of x_n exists, the upper limit is equal to the limit.

For our proof the best definition is the last one although proofs based on the other definitions can also be easily constructed. Here is the actual proof.

Proof. For any $k' > k$,

$$x_{k'} \le \sup_{n>k} x_n.$$

Similarly, for any $k' > k$,

$$y_{k'} \le \sup_{n>k} y_n.$$

Adding two inequalities together, we get

$$x_{k'} + y_{k'} \leq \sup_{n>k} x_n + \sup_{n>k} y_n.$$

Since all $x_{k'} + y_{k'}$ are less then the right-hand side, same is true for the maximum of any number of them, or for their supremum.

$$\sup_{k'>k} (x_{k'} + y_{k'}) \leq \sup_{n>k} x_n + \sup_{n>k} y_n.$$

Taking limit of both sides of the inequality (the limits exist!), we get

$$\lim_{k\to\infty} \left(\sup_{k'>k} (x_{k'} + y_{k'}) \right) \leq \lim_{k\to\infty} \left(\sup_{n>k} x_n \right) + \lim_{k\to\infty} \left(\sup_{n>k} y_n \right),$$

which produces the desired result, according to the last definition. □