Sheet 4

Example 1. a) Expand $f(x) = x^2$ in a Fourier series on $[-\pi, \pi]$. Use the even-odd idea to reduce your work.

Create the directory sheet04 and make it your working directory.

b) In this part, you will plot $f(x)$ and the 4^{th} partial sum of the Fourier series on the same graph. Proceed as follows.

Clear the memory and enter the nth term:

```
clear
nth_term = @ (n,x) whatever you found in part a;
```

There is a little trouble here because this is valid only for $n \geq 1$ and you should have had an a_0 term. So add in the line

```
zeroth_term = whatever you found in part a;
```

Next give a `linspace` for plotting:

```
x = linspace(-pi,pi);
```

Give the partial sum you want (4 in this case):

```
want_terms = 4;
```

and now find the 4^{th} partial sum:

```
for n = 1:want_terms
    terms(n,:)  = nth_term(n,x);
end
y = zeroth_term + sum(terms);
```

The `for` statement computes the n^{th} term for each value of `x` and `y` gives the 4^{th} partial sum for each value of `x`. Now plot everything on the same graph:

```
fcn = x.∧2;
plot(x,y,'r',x,fcn,'k','linewidth',2)
legend('n = 4','x∧2')
```

Repeat for the 8^{th} partial sum—all you need to do is change the `want_terms = 4`. Save your work as ex1b.m.

c) Save ex1b.m as ex1c.m and modify it to plot $f(x)$ and the 4^{th} partial sum on the same graph for $-3\pi \leq x \leq 3\pi$. What did you see? Can you explain it?

d) For $-3\pi \le x \le 3\pi$, plot the periodic extension of $f(x)$ and the 4^{th} partial sum on the same graph. Save this as ex1d.m

Example 2. a) Consider the function

$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi \le x \le 0 \\ 1 & \text{if} \quad 0 < x \le \pi \end{cases}$$

Expand $f(x)$ in a Fourier series on $[-\pi, \pi]$—use paper and pencil. Use the even-odd idea to reduce your work.

b) Here's how to enter a function given in pieces, say $g(x) = \begin{cases} a & \text{if} \quad -1 \le x \le 0 \\ b & \text{if} \quad 0 < x \le 1 \end{cases}$:

```
g = @ (x) a*((-1<=x)&(x<=0))+b*((0<x)&(x<=1))
```

Then when you want values of `g`, just type `g(x)`.

You are going to plot $f(x)$ and the n^{th} partial sum of the Fourier series on $[-\pi, \pi]$ for various choices of n to see how many terms are needed to get a good approximation. Here is what to do: Your Fourier series should have had only sine terms. You will modify the ex1b.m file you saved above (save it as ex2b.m). The major changes will be

- There is no a_0 so you will set `zeroth_term = 0`
- Change `want_terms` to various values
- To plot everything on the same graph, you will replace the statements

  ```
  fcn = x.∧2;
  plot(x,y,'r',x,fcn,'k','linewidth',2)
  ```

  with the statements

  `f =` model this on the way you were told to enter g above
  ```
  fcn = f(x);
  plot(x,y,'r',x,fcn,'k','linewidth',2)
  ```

c) For $-3\pi \le x \le 3\pi$, plot the periodic extension of $f(x)$ and the n^{th} partial sum for the choice of n you made in the previous part, both on the same graph.

Example 3. Expand the function $f(x) = x^3$ in a Fourier series on $[-\pi, \pi]$. This is an odd function, and so all the a_n's will be zero. Rather than integrate by hand (it's nasty), let us have Matlab help. Save a new worksheet as ex3.m. At the beginning of the file, type

```
clear
f = @ (x) x.∧3;
syms x n;
b_n = int(f(x)*sin(n*x),x,-pi,pi)/pi
```

and execute the file. This will tell you the Fourier coefficient b_n in the command window. Highlight this value—it will be a big mess that you don't want to type yourself. Instead, copy it and paste it into the edit window underneath the

```
b_n = int(f(x)*sin(n*x),x,-pi,pi)/pi
```

line you just typed. Then at the beginning of this pasted line insert

```
nth_term = @ (n,x)
```

and at the end add

```
*sin(n*x);
```

so that now the line will read

```
nth_term = @ (n,x) a huge mess *sin(n*x);
```

Copy and paste the appropriate lines from ex1b.m in order to plot $f(x)$ and the 15^{th} partial sum together for $x \in [-\pi, \pi]$. The only other modification you should make is in the `y = zeroth_term + sum(terms)` line. Change it to `y = double(zeroth_term + sum(terms))`. This tells Matlab to evaluate this as a decimal—if you don't do this, you might have trouble generating the plot.

Example 4. Modifying preceeding example, plot $f(x) = 1-(x-1)^2$ and the 4^{th} partial sum together for $x \in [-\pi, \pi]$. The added complication is that no even-odd type simplification occurs in this one. Thus your `nth_term` will have sines and cosines. Also, to avoid trouble you should enter the `zeroth_term` as `zeroth_term = int(f(x),x,-pi,pi)/(2*pi);` Try different n^{th} partial sums. Save your work as ex4.m.

Example 5. Using the method of the preceeding example, plot

$$f(x) = \begin{cases} (\pi + x)^7 & \text{if} \quad -\pi \le x \le 0 \\ (\pi - x)^7 & \text{if} \quad 0 < x \le \pi \end{cases}$$

and the 5^{th} partial sum together for $x \in [-\pi, \pi]$. Here there is even-odd simplification, but the complication is that $f(x)$ is in two pieces. Here are the major changes you should make to the model ex4.m file you created above:

- Leave off the definition of `f` in the line just after the `clear` statement.
- When you compute the `a_n`, you will need to do two integrals: one integral over $[-\pi, 0]$ and another over $[0, \pi]$. In place of the `f(x)`, enter the specific expression for f in each `int` statement. Do a similar thing for the `zeroth_term`

- Replace the lines after the `y = double(zeroth_term + sum(terms))` but before the `plot` statement with the lines

  `f = @ (x)` put the appropriate expression for f here`;`
  `fcn = f(x);`

Try other partial sums.

Sheet 4: Further Exercises

Example 5. Find the Fourier Series on $[-\pi, \pi]$ of the function

$$f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$$

Answer:

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos(nx) - \frac{(-1)^n}{n} \sin(nx) \right]$$

Example 6. Find the Fourier Series on $[-\pi, \pi]$ of the function

$$f(x) = \begin{cases} -\pi & -\pi \leq x \leq 0 \\ 0 & 0 < x \leq \pi \end{cases}$$

Answer:

$$-\frac{\pi}{2} + 2\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

Example 7. If F is $2\pi-$ periodic and c is any real number, show that

$$\int_{-\pi}^{-\pi+c} F(x)\, dx = \int_{\pi}^{\pi+c} F(x)\, dx$$

HINT: Make a $u-$substitution $x = t - 2\pi$.