Sheet 5

Example 1. a) Express the following in the form $a + ib$:

$$(4-i)^2; \qquad (5+i)(2-i); \qquad (2-i)^3; \qquad \frac{1}{1+i}; \qquad \frac{5i+1}{3i-2}; \qquad \frac{1}{2+i} - \frac{i}{i-1};$$

Hint: To express a fraction $\frac{z_1}{z_2}$ in the form $a+ib$, multiply top and bottom by $\bar{z}_2$.

b) For $z = a + ib$, verify the identity $z\bar{z} = |z|^2$.

c) Using Euler's formula, verify the identity $\frac{d}{dt}e^{it} = ie^{it}$ by computing both sides and checking that they match.

d) Using Euler's formula in the expression for β_n, for $n \geq 0$, find β_n in terms of a_n and b_n.

e) For $n \geq 1$, use the relationship between β_n and β_{-n} derived in class along with part c) to find β_{-n} in terms of a_n and b_n.

Example 2. Expand the function $f(x) = e^{-x}$ in a complex Fourier series on $[-\pi, \pi]$. The non-complex Fourier series would be hard to work out because you would have to remember how to integrate $e^{-x}\cos(nx)$ and $e^{-x}\sin(nx)$—they require a trick. But the complex coefficients are very easy to find by hand.

Create the directory sheet05 and make it your working directory.

Example 3. a) Plot the function $f(x) = e^{-x}$ and its complex Fourier series approximation $\sum_{n=-8}^{8} \beta_n e^{inx}$ on $[-\pi, \pi]$. Play around with the choice of the cutoff N.

Here is the code to use:

```
x = linspace(-pi,pi);
y = zeros(size(x));
N = 8;
for n = -N:N,
    betan = your answer from question 2
    y = y + betan*exp(i*n*x);
end
plot(x, y, 'r-', x, exp(x), 'k-','linewidth',2);
```

b) For $-3\pi \leq x \leq 3\pi$, plot the periodic extension of $f(x)$ and the partial sum $\sum_{n=-15}^{15} \beta_n e^{inx}$, both on the same graph. Try other choices of n.

Example 4. a) In this example, you are going to compute numerically the complex Fourier series of $f(x) = x^2$ and generate plots. We can do the integrals by hand (try it! integrate by parts twice), or evaluate them symbolically using `int` function. But instead we will use numerical integration to approximate the coefficients. The default precision is usually good enough unless N is too large. The relevant command is `quad`. So modify your code above by inserting the lines below in appropriate places.

```
f = @ (x) x.∧2;
betan = quad( @(x) f(x) .* exp(-i*n*x), -pi, pi) / (2*pi);
```

Modify the `plot` command accordingly. Play around with N to seee how precision improves.

b) Plot the periodic extension of f and its complex Fourier Series on $[-3\pi, 3\pi]$.

Example 5. a) Plot the function $f(x) = -x^3 - x^5$ and its partial complex Fourier series $\sum_{n=-8}^{8} \beta_n e^{inx}$ on $[-\pi, \pi]$. Play around with the choice of N.

b) For $-3\pi \leq x \leq 3\pi$, plot the periodic extension of $f(x)$ and the partial sum $\sum_{n=-8}^{8} \beta_n e^{inx}$, both on the same graph. Try other choices of N.

Sheet 5: Further Exercises

Example 6. Plot the function

$$f(x) = \begin{cases} -1 - x^2 & \text{if} \quad -\pi \leq x \leq 0 \\ 1 + x^2 & \text{if} \quad 0 < x \leq \pi \end{cases}$$

and its partial complex Fourier series $\sum_{n=-8}^{8} \beta_n e^{inx}$ on $[-\pi, \pi]$. Try other partial sums as well as the periodic extension to the interval $[-3\pi, 3\pi]$.

Example 7. Derive the trig identities given in class using Euler's formula. HINT: expand both sides of the equations

$$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$$

and

$$e^{i(\alpha-\beta)} = e^{i\alpha} e^{-i\beta}$$

using Euler's formula. Note there is no need to perform a division.

Example 8. If f is real-valued and even on the interval $[-\pi, \pi]$, show that the complex Fourier coefficients are real.