

## Sheet 13

Create the directory sheet13 and make it your working directory.

**Example 1.** In some situations, it is important to find out *when* certain events occur: a voltage spike, the onset of heart arrhythmia, a light flash.

a) Go to my homepage and save the file data8.csv to your working directory. Load it (see Sheet 7, Example 6 if you don't remember how) and plot to see it is a signal with a spike. Recall the basic philosophy of Fourier analysis: take the Discrete Fourier Transform (DFT) and work with the Fourier coefficients. By including enough Fourier coefficients in the Fourier series, the series will recover the spike. See how many terms you need to retain—use Example 3 on Sheet 7 as a model for what to do. But note that in contrast, this time you want to include enough terms to *recover* the spike, not filter it out. So in this sense, Fourier analysis can recognize that there is a spike in the signal.

b) However, Fourier analysis cannot give any information about *when* the spike occurred. To understand this, note that the spike is a high frequency event (it happens very quickly), so it will need lots of high frequency sines and cosines to represent it. Thus we expect to see a coefficient  $\beta_n$  in the complex Fourier series that has a relatively large magnitude. In other words, a plot of the magnitude of the complex Fourier coefficients will have a hump that corresponds to the spike.

Download the file data9.csv from my homepage and load it. Plot it to see it too has a spike, but now it is located at a different time when compared to data8.csv. In fact, without the spikes, the signals are exactly the same. Plot the magnitudes of the complex Fourier coefficients for this signal and the one from part a) on the same graph as follows. Assuming the data from the signal in part a) is  $y_1$ , the new data is  $y_2$  and the time variable is  $t$ , enter

```
fy1 = fft(y1);
fy2 = fft(y2);
n = [0:length(t)-1]
hold on
plot(n,abs(fy1),'o','color','r')
plot(n,abs(fy2),'x')
hold off
```

Include an `axis` statement to zoom in on the action near the time axis. You should see that the plots are exactly the same and you should see a hump in the plot corresponding to the large high frequency coefficient giving the spike. Remember that the spikes occurred at a different times, but there is no way you can just look at the Fourier coefficients of the two signals to tell *when* the spikes occur. There is a subtle point here: Of course you can reconstruct the signal from the coefficients and see where the spike occurred, but the idea of Fourier analysis is to look only at the Fourier coefficients to glean information about the signal. Think of automating the task: a computer would compute the Fourier coefficients and run various algorithms on the coefficients to analyze the signal. But a computer cannot look at a plot to see where a spike occurred.

Conclusion: Looking at the complex Fourier coefficients (i.e., Fourier analysis) does not give any information about *when* events occur.

c) On the other hand, Haar wavelets do a great job of telling when the spike occurs in the signal. The data set in part a) has  $2^9$  points (the command `length(t)` in the command window will tell you this), so the signal is an element of  $V_9$ . Perform a level one decomposition (i.e.,  $V_9 = W_8 \oplus V_8$ ) and plot the absolute value of the level one details  $D_1$  (i.e., the  $W_8$  component) and the signal on the same graph—see Sheet 11, Example 1. Note your plot statement should have `abs(Dn)` in place of `Dn`.

You should see the largest value of the level one details occurs at the same time as the spike!!

Now let us use the level one coefficients Matlab computes to find out when the spike occurred. Here is the basic idea: Suppose the time interval of the signal is from  $t = 0$  to  $t = a$ . The level one details are exactly the  $W_8$  component and an element of  $W_8$  is a sum of spikes

$$b_1\psi(2^8x - m_1a) + \cdots + a_p\psi(2^8x - m_pa).$$

Each of the spikes is  $\frac{a}{2^8}$  units wide. What Matlab does is write a function in  $W_8$  as

$$b_1\psi(2^8x) + b_2\psi(2^8x - a) + b_3\psi(2^8x - 2a) + \cdots + b_{2^8}\psi(2^8x - 2^8a + a)$$

and outputs a list consisting of all the  $b_\ell$ 's; here some of the  $b_\ell$ 's might be 0. The important thing is that  $b_1$  corresponds to the coefficient of the spike on the interval  $[0, \frac{a}{2^8}]$ ,  $b_2$  corresponds to the spike on  $[\frac{a}{2^8}, \frac{2a}{2^8}]$ ,  $b_3$  to  $[\frac{2a}{2^8}, \frac{3a}{2^8}]$ ,  $\dots$   $b_\ell$  corresponds to  $[\frac{(\ell-1)a}{2^8}, \frac{\ell a}{2^8}]$ , etc. Thus if  $|b_5|$  is the maximum of the absolute value of all the  $W_8$  coefficients, then we know the maximum occurred during the time interval  $[\frac{4a}{2^8}, \frac{5a}{2^8}]$ . Thus a good choice for when the spike occurred is the midpoint of this time interval.

Here is how to have Matlab help you find the time of the spike. Assuming your `wavedec` statement was `[C,L] = wavedec...`, assign the vector of coefficients to the variable `coef`:

```
coef = detcoef(C,L,1);
```

(the last 1 is for level one decomposition—it will give the  $W_8$  coefficients in this case). Now find the coefficient with the biggest absolute value:

```
[ d, m ] = max(abs(coef))
```

Here `d` will be the maximum value and `m` will be its location; that is,  $d = \max_{\ell} |b_\ell| = |b_m|$ . Now you can find when the spike occurred, once you figure out how long the time interval is (i.e., `max(t) - min(t)`). **Answer:** .3559

**Example 2:** Repeat the analysis of part c) from the preceding example on the signal from part b) of the example and find when that spike occurred. **Answer:** 5.9273

**Example 3:** Download the file `data10.csv` from my homepage—it is a signal with a jump. Use a level 1 Haar decomposition to find when the jump occurred. The true value is 0.41. How did you do? Repeat for the files `data11.csv` and `data12.csv`. The correct answers for those files are 0.89 and 0.29, respectively.

### Sheet 13: Further Exercises

**Example 4:** Let  $f(t) = 1$  for  $-\pi \leq t \leq \pi$ , zero otherwise. Find the Fourier transform of  $f$ . HINT: Euler's formula, even-odd.

**Answer:**  $\sqrt{\frac{2}{\pi}} \frac{\sin(\lambda\pi)}{\lambda}$

**Example 5:** Let  $f(t) = \cos(nt)$  for  $-\pi \leq t \leq \pi$ , zero otherwise. Find the Fourier transform of  $f$ . HINT: Euler's formula, trig identities, even-odd.

**Answer:**  $\sqrt{\frac{2}{\pi}} \frac{\lambda \sin(\lambda\pi)}{(n^2 - \lambda^2)}$

**Example 6:** Plot the Fourier transform  $\hat{f}(\lambda)$  for  $n = 3$  in the preceding example. I suggest taking  $-10 \leq \lambda \leq 10$ . Notice  $f(t) = \cos(3t)$  is its own Fourier series. Thus all the Fourier coefficients  $a_n$  and  $b_n$  are zero, *except*  $a_3 = 3$ . Is this consistent with the plot?

**Example 7:** a) For  $n = 2$  in Example 5, plot  $f(t)$  and  $\hat{f}(\lambda)$  on separate graphs, but in the same plot window, using the subplot command. Play around with the  $\lambda$ -range.

b) Repeat for  $n = 4$  and  $n = 6$ .

c) Comments?

**Example 8:** Let

$$f(t) = \begin{cases} t + \pi & \text{if } -\pi \leq t \leq 0 \\ \pi - t & \text{if } 0 < t \leq \pi \\ 0 & \text{otherwise.} \end{cases}$$

Find the Fourier transform of  $f$ . HINT: Euler's formula, even-odd.

**Answer:**  $\sqrt{\frac{2}{\pi}} \frac{(1 - \cos(\lambda\pi))}{\lambda^2}$

**Example 9:** Let  $f(t) = e^{-t}$  for  $t \geq 0$ , zero otherwise. Find the Fourier transform of  $f$  and express it in the form  $a + ib$ . HINT: To make the denominator real, multiply top and bottom by the conjugate of the denominator.

**Answer:**  $\frac{1}{\sqrt{2\pi}} \frac{1}{1 + \lambda^2} - i \frac{1}{\sqrt{2\pi}} \frac{\lambda}{1 + \lambda^2}$

**Example 10:** Let  $f(t) = e^{-t^2}$ . Find the Fourier transform  $\hat{f}(\lambda)$ . HINT: Complete the square, make a (complex) change of variables in the resulting integral and use the fact that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

This is *extremely* important in probability theory—it has to do with the normal distribution.

**Answer:**  $\frac{1}{\sqrt{2}} e^{-\lambda^2/4}$

**Example 11:** Verify the identity in the preceding exercise. HINT: Write

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx dy\end{aligned}$$

and convert to polar coordinates.

**Example 12:** Let  $L$  be the line through the point  $(1, 1)$ , perpendicular to  $\vec{n} = \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ . Evaluate the Randon transform

$$\int_L f(\vec{x}) |d\vec{x}|,$$

where

$$f(\vec{x}) = \frac{x - y}{[(x - 1)^2 + (y - 1)^2 + 1]^2} \quad \text{and} \quad \vec{x} = (x, y).$$